ON A PIERI–LIKE RULE FOR THE PETRIE SYMMETRIC FUNCTIONS

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ABSTRACT. A k-ribbon tiling is a decomposition of a connected skew diagram into disjoint ribbons of size k. In this paper, we establish a connection between a subset of k-ribbon tilings and Petrie symmetric functions, thus providing a combinatorial interpretation for the coefficients in a Pieri-like rule for the Petrie symmetric functions due to Grinberg (Algebr. Comb. 5 (2022), no. 5, 947-1013). This also extends a result by Cheng *et al.* (Proc. Amer. Math. Soc. 151 (2023), no. 5, 1839-1854). As a bonus, our findings can be effectively utilized to derive certain specializations.

1. INTRODUCTION AND BACKGROUND

Petrie symmetric functions were introduced by Doty–Walker [6] and Bazeniar–Ahmia–Belbachir [2] when they studied a class of truncated tensor products of representations of the general linear group, and an extension of Pascal triangles, respectively. Very recently, Fu–Mei [8] and Grinberg [9] independently investigated a series of nice properties of Petrie symmetric functions. The former was motivated by its natural unification of elementary symmetric functions and complete homogeneous symmetric functions. The latter was inspired by Liu and Polo's work on the cohomology of line bundles over a flag scheme [13, 14].

The *Petrie symmetric functions* $G(k, m)$ of degree m are defined by

$$\sum_{m=0}^{\infty} G(k,m)(x)z^m = \prod_{i=1}^{\infty}\left(\sum_{j=0}^{k-1}(x_i z)^j\right) = \prod_{i=1}^{\infty}\frac{1-(x_i z)^k}{1-x_i z} = \prod_{i=1}^{\infty}\prod_{j=1}^{k-1}\left(1-\omega^j x_i z\right), \tag{1.1}$$

where $\omega = e^{\frac{2\pi i}{k}}$ is the primitive k-th root of unity.

By the first equality of (1.1) and the generating function of partitions, one gets an equivalent definition of $G(k, m)(x)$ in terms of monomial symmetric functions.

$$G(k,m)(x) = \sum_{\substack{\lambda\vdash m\\ \lambda_1<k}} m_\lambda(x). \tag{1.2}$$

In particular, $G(2, m)(x) = e_m(x)$, $G(k, m)(x) = h_m(x)$ if $k > m$ and $G(k, m)(x) = h_m(x) - p_m(x)$ if $k = m$. A Pieri-like rule for the Petrie symmetric functions was proved by Grinberg [9]. Throughout the paper, we define $\chi(A) = 1$ if the statement A is true; otherwise $\chi(A) = 0$.

Theorem 1.1. [9] *For $k \in \mathbb{N}^+$, $m \in \mathbb{N}$ and any partition μ,*

$$G(k,m)(x)s_\mu(x) = \sum_{\lambda\vdash m+|\mu|} \mathrm{Pet}_k(\lambda,\mu)s_\lambda(x), \tag{1.3}$$

where $\mu \subseteq \lambda$ and $\mathrm{Pet}_k(\lambda,\mu) = \det_{1\le i,j\le\ell(\lambda)}\left(\chi(0 \le \lambda_i - \mu_j - i + j < k)\right)$.

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The integers $\mathrm{Pet}_k(\lambda,\mu)$ are called *k-Petrie numbers*. Grinberg further showed that $\mathrm{Pet}_k(\lambda,\mu)$ have three possible values $0,-1,1$. Our main result interprets the k-Petrie number as a finite product associated to a proper k-ribbon tiling (see Theorem 1.4).

Before stating the main result, we first introduce some necessary definitions and notations. For any skew diagram $\mathcal{K}$, the size of $\mathcal{K}$, denoted by $|\mathcal{K}|$, is the number of boxes of $\mathcal{K}$.

Definition 1.2 (proper k-ribbon tiling)**.** A k-ribbon tiling $\Theta=(\Theta_1,\dots,\Theta_m)$ of a skew diagram $\mathcal{K}$ is a sequence of ribbons Θ_i of size k whose disjoint union is exactly $\mathcal{K}$, denoted by $\dot\cup_{i=1}^m\Theta_i=\mathcal{K}$. Let $\mathcal{K}=\lambda/\mu$ and ν be any partition such that $\mu\subseteq\nu\subseteq\lambda$, we call a tiling (Θ,ν) of λ/μ *proper* if Θ is a k-ribbon tiling of λ/ν satisfying the following two conditions:

- **(i)** ν/μ is a horizontal strip, that is, no two boxes are in the same column;
- **(ii)** the starting box of each k-ribbon in Θ is the leftmost box of a row of λ/ν.

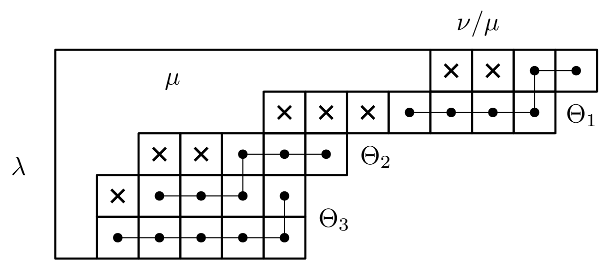

FIGURE 1. Let $\lambda=(13,12,7,6,6)$, $\mu=(9,5,2,1,1)$ and $\nu=(11,8,4,2,1)$, an even proper 6-ribbon tiling $(\Theta,\nu)\in\mathcal{E}_8^6(\lambda,\mu)$ is displayed where every box of ν/μ and of λ/ν is marked by a cross $\times$ and a black dot $\bullet$ in the center respectively.

Remark 1.3. *The k-ribbon tilings are used to describe the plethystic Murnaghan-Nakayama rule (see for instance* [3, 5, 7, 17]*) and related to specializations of Schur functions at primitive roots of unity (see for instance* [15, p. 91]*).*

Let $r(\mathcal{K})$ be the number of rows of $\mathcal{K}$, we say that a proper tiling (Θ,ν) is *odd* if $r(\Theta):=r(\Theta_1)+\cdots+r(\Theta_m)$ is odd; and *even* otherwise.

Let $\mathcal{O}_n^k(\lambda,\mu)$ (resp. $\mathcal{E}_n^k(\lambda,\mu)$) be the set of odd (resp. even) proper k-ribbon tilings of λ/μ with $|\nu/\mu|=n$. Set $\mathcal{D}_n^k(\lambda,\mu)=\mathcal{O}_n^k(\lambda,\mu)\,\dot\cup\,\mathcal{E}_n^k(\lambda,\mu)$. Then $|\lambda|-|\mu|-n$ must be a multiplier of k unless $\mathcal{D}_n^k(\lambda,\mu)=\varnothing$. See Figure 1 for an example.

We are now ready to state our main result:

Theorem 1.4. *Let μ,λ be two partitions such that $\mu\subseteq\lambda$.*

(1) If λ/μ is connected, then either $|\mathcal{D}_n^k(\lambda,\mu)|\in\{0,1\}$ or $|\mathcal{D}_n^k(\lambda,\mu)|$ is even and $|\mathcal{O}_n^k(\lambda,\mu)|=|\mathcal{E}_n^k(\lambda,\mu)|$.

(2) $\mathrm{Pet}_k(\lambda,\mu)=0$ unless λ/μ has exactly one unique proper k-ribbon tiling and every row of λ/μ has size less than k. In this case, let (Θ,ν) with $\Theta=(\Theta_1,\cdots,\Theta_m)$ be the unique k-ribbon tiling of λ/μ, then

$$\mathrm{Pet}_k(\lambda,\mu)=\prod_{i=1}^m(-1)^{r(\Theta_i)}. \tag{1.4}$$

Equation (1.4) recovers and extends the combinatorial description of $\mathrm{Pet}_k(\lambda, \varnothing)$ by Cheng *et al.* [4]. Their proof employs Grinberg's combinatorial formula of $\mathrm{Pet}_k(\lambda, \varnothing)$ and a coding from bounded partitions to abacuses with bounded vertical runners. Our proof consists of an equality for the number of proper tilings, the plethystic Murnaghan-Nakayama rule and applications of the λ-ring theory. Our method provides a different and alternative approach to study Petrie symmetric functions, and is also feasible for modular Schur functions [6]. Especially we would like to draw attention to the λ-ring or vertex algebraic approach which helps unveiling intrinsic relations between Petrie symmetric functions and Schur symmetric functions.

The rest of the paper is organized as follows. In Section 2, we review some terminologies regarding tilings, symmetric functions and the λ-ring theory. In Section 3, we discuss the number of proper tilings of a connected skew diagram, which establishes the first part of Theorem 1.4. Section 4 is devoted to proving the second part of Theorem 1.4 and two specializations that are consequences of Theorem 1.4 (see Corollary 4.5 and Corollary 4.7).

2. Preliminaries

This section presents some terminologies on skew partitions, symmetric functions and λ-rings. For a complete description, we refer to the books [11] and [15, Section 1].

2.1. **Skew diagrams, ribbons and proper tilings.** A *partition* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of n, denoted by $\lambda \vdash n$, is a sequence of positive integers such that $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i < k$. Each λ_i is called a *part* of λ. We use $\ell(\lambda)$ and $|\lambda|$ to denote the *length* and the *size* of λ, that is, $\ell(\lambda) = k$ and $|\lambda| = n$.

A partition is identified with its *Young diagram*, which is the left-justified array of squares consisting of λ_i squares in the i-th row from top to bottom for $1 \leq i \leq \ell(\lambda)$. Here all rows (resp. columns) of a skew diagram are numbered from top to bottom (resp. from left to right). A box z has *coordinate* (i, j) if it is located on the i-th row and the j-th column and we say that the box $z = (i, j)$ has *content* $c(z) = j - i$. Two boxes are said to be on the same diagonal if they have the same content. The *conjugate* of λ, denoted by λ', is obtained from λ by transposing the diagram λ along the diagonal of content zero.

For two partitions λ and μ, we write $\mu \subseteq \lambda$ if the Young diagram of μ can be drawn inside that of λ, i.e., $\mu_i \leq \lambda_i$ for all $i \geq 1$. In this case the set-theoretic difference $\theta = \lambda - \mu$ is called a *skew diagram*, denoted by $\theta = \lambda/\mu$ and $|\theta| = |\lambda| - |\mu|$ is the size of λ/μ. The *height* $ht(\lambda/\mu)$ of a connected skew diagram λ/μ is defined to be one less than the number of rows of λ/μ, that is, $ht(\lambda/\mu) = r(\lambda/\mu) - 1$.

A *path* in a skew diagram θ is a sequence $x_0, x_1, \cdots, x_n$ of squares in θ from bottom-left to top-right such that x_{i-1} and x_i have a common edge, for $1 \leq i \leq n$. A subset ξ of θ is *(edgewise) connected* if any two squares in ξ are connected by a path in ξ. The connected components are the maximal connected subdiagrams of θ.

A skew diagram λ/μ is a *vertical* (resp. *horizontal*) *strip* if each row (resp. column) contains at most one cell. Note that a vertical or horizontal strip is connected only if it is a column or a row. A *ribbon* is a connected skew diagram without 2×2 block of boxes. A *k-ribbon* is a ribbon of size k. Geometrically a k-ribbon is a string of k cells; see for instance Figure 1. Given a ribbon, the leftmost box of the bottom row (also the southwesternmost box) is called *the starting box*, denoted by s; and the rightmost box of the top row (also the northeasternmost box) is called *the ending box*, denoted by t.

We introduce dual proper tilings by their conjugacy.

Definition 2.1 (dual proper tiling)**.** For a skew diagram λ/μ, let ν be any partition such that $\mu \subseteq \nu \subseteq \lambda$. A *dual proper tiling* (Θ, ν) of λ/μ is a k-ribbon tiling with $\Theta = (\Theta_1, \ldots, \Theta_m)$ satisfying

- **(i$'$)** ν/μ is a vertical strip;
- **(ii$'$)** the ending box of each Θ_i is the topmost box of a column of λ/ν.

Equivalently, the pair (Θ, ν) is a dual proper tiling of λ/μ if and only if (Θ', ν') is a proper tiling of λ'/μ', where $\Theta' = (\Theta'_1, \ldots, \Theta'_m)$.

2.2. **Symmetric functions and λ-rings.** Let Λ be the ring of symmetric functions in infinite variables $x = (x_1, x_2, \ldots)$ over the rational field $\mathbb{Q}$. The linear bases of Λ are indexed by partitions. Let $\mathcal{P}$ denote the set of all partitions. For every $\lambda \in \mathcal{P}$, let $p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x)\cdots p_{\lambda_{\ell(\lambda)}}(x)$ with

$$p_r(x) = \sum_{i=1}^{\infty} x_i^r$$

being the *power-sum symmetric functions*. The set of $p_\lambda(x)$ for all $\lambda \in \mathcal{P}$ forms a $\mathbb{Q}$-basis of Λ. If we consider the space Λ over the integer ring $\mathbb{Z}$, then Λ has the following bases indexed by partitions. A *weak composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n is a sequence of nonnegative integers whose sum equals n. The *monomial symmetric function*

$$m_\lambda(x) = \sum_{\alpha} x^\alpha = \sum_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

which is summed over all weak compositions α that are permutations of parts of λ and infinitely many zeros. The *elementary symmetric function* is defined by $e_\lambda(x) = e_{\lambda_1}(x)\cdots e_{\lambda_{\ell(\lambda)}}(x)$ where

$$e_r(x) = m_{(1^r)}(x) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

The *complete symmetric function* is $h_\lambda(x) = h_{\lambda_1}(x)\cdots h_{\lambda_{\ell(\lambda)}}(x)$ with

$$h_r(x) = \sum_{\lambda \vdash r} m_\lambda(x) = \sum_{i_1 \le \cdots \le i_r} x_{i_1} \cdots x_{i_r}.$$

A *semi-standard Young tableau* of a skew shape is a filling of positive integers to the Young diagram of λ such that each cell is filled with exactly one integer, entries along each row are weakly increasing from left to right, and entries along each column are strictly increasing from top to bottom. The *Schur function* is given by

$$s_\lambda(x) = \sum_T x^T = \sum_T x_1^{T_1} x_2^{T_2} \cdots$$

where the sum is ranged over all semi-standard Young tableaux T of shape λ and T_i counts the occurrences of entry i in T. In addition, the elementary symmetric functions and the complete homogeneous symmetric functions are characterized by their generating function formulas:

$$E(z) := \sum_{n \ge 0} e_n(x) z^n = \prod_{i \ge 1} \left(\sum_{0 \le j \le 1} (x_i z)^j \right) = \prod_{i \ge 1} (1 + x_i z), \tag{2.1}$$

$$H(z) := \sum_{n\geq 0} h_n(x)z^n = \prod_{i\geq 1}\left(\sum_{j\geq 0}(x_iz)^j\right) = \prod_{i\geq 1}\frac{1}{1-x_iz}. \tag{2.2}$$

The theory of λ-rings is mainly built upon one of the symmetric functions [11]. A *λ-ring* is a commutative ring $\mathcal{R}$ with identity 1 and with operations $\lambda^i : \mathcal{R} \to \mathcal{R}$ for $i \in \mathbb{N}$, such that for all $x \in \mathcal{R}$, the formal power series

$$\lambda_z(x) = \sum_{i=0}^{\infty} \lambda^i(x)z^i = \lambda^0(x) + \lambda^1(x)z + \lambda^2(x)z^2 + \cdots \tag{2.3}$$

satisfies $\lambda^0(x) = 1$, $\lambda^1(x) = x$,

$$\lambda_z(x+y) = \lambda_z(x)\,\lambda_z(y) \quad\text{and}\quad \lambda_z(xy) = \sum_{i=0}^{\infty}\lambda^i(x)\lambda^i(y)z^i. \tag{2.4}$$

The first equality of (2.4) gives $\lambda_z(0) = 1$ and $\lambda_z(x) = \lambda_z(-x)^{-1}$ by taking $y = 0$ and $y = -x$, respectively. In fact, (2.4) guarantees that λ_z is a homomorphism from $\mathcal{R}$ to the ring $\mathcal{R}[[z]]$ of formal power series with constant term 1. Given two λ-rings $\mathcal{R}_1, \mathcal{R}_2$, a *$\lambda$-homomorphism* $\psi : \mathcal{R}_1 \to \mathcal{R}_2$ is a homomorphism of rings such that

$$\psi(\lambda^r(x)) = \lambda^r(\psi(x)) \tag{2.5}$$

for all $x \in \mathcal{R}_1$ and $r \in \mathbb{N}$.

Consider the ring Λ of symmetric functions over $\mathbb{Q}$, which is a free λ-ring on one generator, i.e., $\Lambda = \mathbb{Q}[e_1, e_2, \cdots]$ with $\lambda^r(e_1) = e_r$. We define $\sigma^r(x) := (-1)^r\lambda^r(-x)$ and

$$\sigma_z(x) := \sum_{i=0}^{\infty}\sigma^i(x)z^i = \sum_{i=0}^{\infty}\lambda^i(-x)(-z)^i = \lambda_{-z}(-x).$$

For any λ-ring $\mathcal{R}$, there is a unique λ-homomorphism $\psi : \Lambda \to \mathcal{R}$, by which e_1 is mapped to x and furthermore

$$e_r \mapsto \lambda^r(x); \tag{2.6}$$
$$h_r \mapsto \sigma^r(x); \tag{2.7}$$
$$E(z) \mapsto \lambda_z(x); \tag{2.8}$$
$$H(z) \mapsto \sigma_z(x). \tag{2.9}$$

These relations are essentially consequences of (2.1), (2.2) and (2.5). Let $s^{\lambda/\mu}(x)$ be the image of $s_{\lambda/\mu}$ under ψ, i.e., $s^{\lambda/\mu}(x) := \psi(s_{\lambda/\mu})$, we have

$$s^{\lambda/\mu}(-x) = (-1)^{|\lambda/\mu|}s^{\lambda'/\mu'}(x). \tag{2.10}$$

This is the *duality rule* for skew Schur functions in the λ-ring notations, which follows from (2.6), (2.7), the relation $\sigma^r(x) = (-1)^r\lambda^r(-x)$ and the Jacobi-Trudi identity; see also [15, p. 43]. Let us recall the *Jacobi-Trudi identity*: For any partitions μ, λ such that $\mu \subseteq \lambda$ and $\ell(\lambda) = n$, we have

$$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j})_{i,j=1}^{n} \tag{2.11}$$

where $h_0 = 1$ and $h_k = 0$ for $k < 0$. Let $y = (y_1, y_2, \ldots)$, define $xy = (x_i y_j)_{i,j\geq 1}$ and $x + y = (x_i, y_j)_{i\geq 1, j\geq 1}$, then the *sum rule* is given by

$$s^{\lambda/\mu}(x+y) = \sum_{\mu\subseteq\nu\subseteq\lambda} s^{\lambda/\nu}(x)s^{\nu/\mu}(y), \tag{2.12}$$

as a result of the Littlewood-Richardson rule (see [15, (5.10)]).

3. On the number of proper tilings

This section is devoted to proving the first part of Theorem 1.4, that is (1), an equality for the number of proper tilings of a connected skew diagram λ/μ. In what follows, we always assume that λ/μ is connected unless otherwise stated.

For each $\mathcal{D}_n^k(\lambda, \mu)$, let

$$m := \frac{|\lambda| - |\mu| - n}{k},$$

then $\mathcal{D}_n^k(\lambda, \mu) = \varnothing$ if $m \notin \mathbb{N}^+$. Therefore, we only consider the case that $m \in \mathbb{N}^+$ and will establish (1) of Theorem 1.4 by an inductive argument. For this purpose, we need to introduce an involution ω, which is essentially a swap of certain boxes from λ/ν and ν/μ. This is to be proved in Lemma 3.3.

There are two steps to achieve Lemma 3.3. In the first step, we mark the center of each box of a skew diagram by a cross or a dot. In the second step, we introduce a set $\mathcal{R}$ of skew diagrams, consisting of these two kinds of boxes, and define the map $\omega : \mathcal{R} \to \mathcal{R}$. Finally, a close inspection of the construction of ω gives that $\omega^{-1} = \omega$, namely, ω is an involution.

Given a ribbon α of $\bullet$–boxes and a set s of ribbons consisting of $\times$–boxes, we say that s appears to the *front* (resp. *end*) of α if one $\times$–box of s is to the bottom left (resp. top right) of α. Otherwise, s is *below* (resp. *above*) α if one $\times$–box of s is to the *south* (resp. *north*) of α. A schematic diagram for the locations of $\times$–boxes is shown in Figure 2.

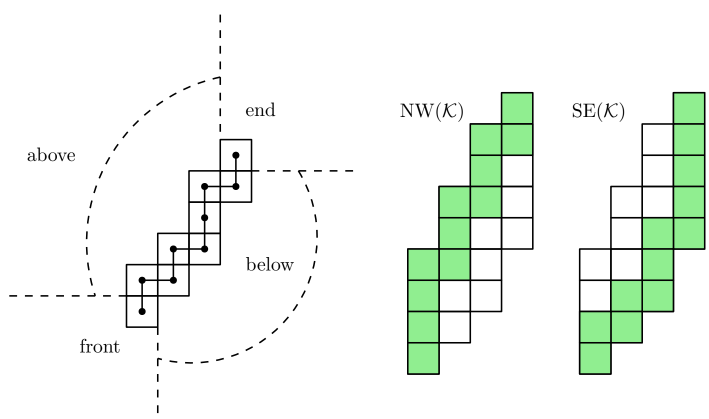

Figure 2. Different positions for $\times$–boxes (left), the northwest ribbon NW($\mathcal{K}$) (middle) and the southeast ribbon SE($\mathcal{K}$) (right) of a skew diagram $\mathcal{K}$.

For a skew diagram $\mathcal{K}$, let SE($\mathcal{K}$) (resp. NW($\mathcal{K}$)) be the southeast (resp. northwest) ribbon of $\mathcal{K}$, which starts from the southwesternmost box of $\mathcal{K}$, traverses the southeast (resp. northwest) border of $\mathcal{K}$, and ends at the the northeasternmost box of $\mathcal{K}$ (see Figure 2).

Definition 3.1. Let $\mathcal{R}$ be the set of pairs (α, s) where α is a ribbon of $\bullet$–boxes and s is a set of ribbons (not necessarily connected) consisting of $\times$–boxes such that

(1) its disjoint union $\alpha \mathbin{\dot\cup} s$ is a connected skew diagram.
(2) s may appear to the front or the end of α, but not both.
(3) s may appear below or above α, but not both.
(4) Let γ_1 (resp. γ_2) be the ribbon that starts from the southwesternmost box (resp. ends at the northeasternmost box) of $\alpha \mathbin{\dot\cup} s$ such that each box is the only box of its diagonal and all boxes are uniformly $\bullet$ or $\times$. Then $|\gamma_1| \leq |\gamma_2|$ if γ_1 is a non-empty ribbon of $\times$–boxes, and $|\gamma_2| \leq |\gamma_1|$ if γ_2 is a non-empty ribbon of $\times$–boxes.

See the first two skew diagrams of Figure 3 as examples from $\mathcal{R}$. We are now in a position to define the map $\omega : \mathcal{R} \to \mathcal{R}$.

Definition 3.2 (The map ω)**.** For a given pair $(\alpha, s) \in \mathcal{R}$ and every $\times$-box of s,

(I) if it is not the only box of its diagonal of $\alpha \mathbin{\dot\cup} s$, then swap it with the unique $\bullet$–box of the same diagonal;
(II) otherwise it is the only box of its diagonal,
   - if it appears to the front of α, that is, γ_1 is a ribbon of $\times$–boxes, then $|\gamma_1| \leq |\gamma_2|$. Let $\gamma \subseteq \gamma_2$ be the ribbon of size $|\gamma_1|$ that has the same ending box of γ_2. Replace each $\times$–box of γ_1 by a $\bullet$–box, and replace each $\bullet$–box of γ by a $\times$–box;
   - otherwise it appears to the end of α, that is, γ_2 is a ribbon of $\times$–boxes, then $|\gamma_2| \leq |\gamma_1|$. Let $\gamma \subseteq \gamma_1$ be the ribbon of size $|\gamma_2|$ that has the same starting box of γ_1. Replace each $\times$–box of γ_2 by a $\bullet$–box, and replace each $\bullet$–box of γ by a $\times$–box.

Define $\omega(\alpha, s) = (\beta, s')$ where s' is the set of $\times$–boxes and β is the set of $\bullet$–boxes after implementing (I) and (II).

We give an example of ω in Figure 3. In the next step, we examine that ω is a well-defined involution on $\mathcal{R}$.

Lemma 3.3. *The map $\omega : \mathcal{R} \to \mathcal{R}$ is an involution by which the number of $\times$–boxes is preserved.*

Proof. We will verify that $(\beta, s') \in \mathcal{R}$ by conditions (1)–(4) of Definition 3.1. Since $\alpha \mathbin{\dot\cup} s = \beta \mathbin{\dot\cup} s'$, condition (1) is clearly satisfied.

The step (I) implies that a $\times$–box of s appears below (or above) α if and only if a $\times$–box of s' appears above (or below) β. Therefore, condition (3) is satisfied by s'.

For the step (II), without loss of generality, suppose that γ_1 is a ribbon of $\times$–boxes from s, then γ_2 must be a ribbon of $\bullet$–boxes as s can not appear both to the front and the end of α by (2). Further, $|\gamma_1| \leq |\gamma_2|$ by condition (4), which means, after applying ω, that the ribbon γ of $\times$–boxes appears to the end of β and no $\times$–box is located to the front of β. This verifies condition (2) for s'.

We next check condition (4) for the pair (β, s'). Let ρ be the ribbon starting from the southwesternmost box of $\alpha \mathbin{\dot\cup} s$ such that every box is the only box of its diagonal. Since γ_1 as a ribbon of $\times$–boxes appears to the front of α, we have $\gamma_1 \subseteq \rho$, thus $|\gamma_1| \leq |\rho|$. It follows that (β, s') has property (4) by noting that the ribbon γ of size $|\gamma_1|$ with $\times$–boxes appears to the end of β. This completes the proof that $(\beta, s') \in \mathcal{R}$.

It is not hard to see that ω is an involution as the construction for each of (I) and (II) is an involution. In addition $|s| = |s'|$, meaning that the number of $\times$–boxes is preserved by ω.

□

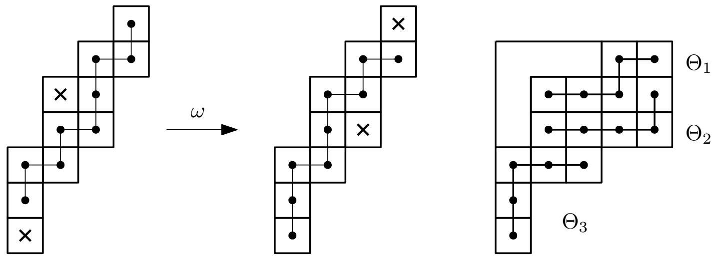

FIGURE 3. An example of the involution ω (left) and a proper tiling (right).

Before we proceed with proving Theorem 1.4, we make an important observation, introduce the concept of undetermined $\times$-boxes and some auxiliary lemmas.

Observation 3.4. Suppose that $\Theta = (\Theta_1, \Theta_2, \dots, \Theta_m)$ is a k-ribbon tiling of λ/ν satisfying condition (ii). Then we can number these m ribbons according to the relation

$$\nu \subseteq \nu \,\dot\cup\, \Theta_1 \subseteq \cdots \subseteq \nu \,\dot\cup\, \Theta_1 \,\dot\cup \cdots \dot\cup\, \Theta_m = \lambda,$$

where Θ_i is to the northeast of Θ_j for any $i < j$, that is, the starting box (resp. the ending box) of Θ_i is to the northeast of that of Θ_j. In terms of contents, it is equivalent to say that the content of the starting box (or the ending box) of Θ_i is strictly greater than that of Θ_j. Figure 3 (right) presents an example for $m = 3$ and $k = 5$.

Note that Θ is uniquely determined by (ii) if each $\times$–box of ν/μ is fixed. Thus there must exist a $\times$–box whose coordinate is not agreed among all proper k-ribbon tilings of $\mathcal{D}^k_n(\lambda, \mu)$ provided that $|\mathcal{D}^k_n(\lambda, \mu)| \geq 2$. We call such $\times$–box *undetermined*; otherwise *determined*. Let ξ be the set of determined $\times$–boxes of λ/μ, then $|\mathcal{D}^k_n(\lambda, \mu)| = |\mathcal{D}^k_{n-|\xi|}(\lambda, \mu \,\dot\cup\, \xi)|$, and it suffices to assume that all $\times$–boxes are undermined.

Lemma 3.5. *If $m = 1$, then $|\mathcal{D}^k_n(\lambda, \mu)| \in \{0, 1, 2\}$. In particular if $|\mathcal{D}^k_n(\lambda, \mu)| = 2$, then $|\mathcal{E}^k_n(\lambda, \mu)| = |\mathcal{O}^k_n(\lambda, \mu)| = 1$.*

Proof. We only need to prove the statement for $m = 1$ and $|\mathcal{D}^k_n(\lambda, \mu)| > 1$. In this case, any proper k-ribbon tiling (Θ_1, ν) contains only one k-ribbon $\Theta_1 = \lambda/\nu$. By Definition 1.2, all undetermined $\times$–boxes are located along the NW(λ/μ). Furthermore, any $\times$–box from ν/μ can not stay above Θ_1 because otherwise it would be determined, a contradiction. It follows that $\times$–boxes appear either to the front or at the end of Θ_1 such that $\Theta_1 \dot\cup (\nu/\mu) = \lambda/\mu$ is a ribbon. This further implies that the top row of λ/μ is either occupied by $\times$–boxes or by $\bullet$–boxes. Therefore, there are at most two possible proper tilings, respectively formed by (Θ_1, ν) and $\omega(\Theta_1, \nu)$. On the other hand, we assume $|\mathcal{D}^k_{n,1}(\lambda, \mu)| \geq 2$, so $|\mathcal{D}^k_{n,1}(\lambda, \mu)| = 2$. Let $\omega(\Theta_1, \nu) = (\Theta'_1, \nu')$. In particular, $r(\Theta_1)$ and $r(\Theta'_1)$ differ by one because exactly one of ν, ν' contains the top row of λ/μ, that is, exactly one of the two proper tilings is odd (or even). This finishes the proof and we refer to Figure 4 for an example. □

For every $(\Theta, \nu) \in \mathcal{D}^k_n(\lambda, \mu)$, let Θ_1 be the k-ribbon from Θ whose ending box is the northeasternmost box of λ/ν as in Observation 3.4. Let α be the k-ribbon along NW(λ/μ) whose ending box is the northeasternmost box of λ/μ. Let u denote the row of boxes appearing to the west of α (u can be empty). We label α and u on the left of Figure 4.

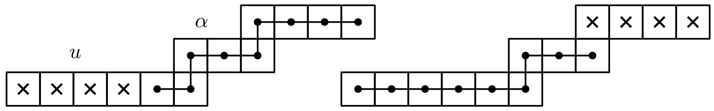

FIGURE 4. Two proper 9-ribbon tilings where α is marked on the left and $u = \times\times\times\times$.

Lemma 3.6. *If $u \neq \varnothing$ and $m \geq 2$, then*

1. *For every $(\Theta, \nu) \in \mathcal{D}_n^k(\lambda, \mu)$, either u is a row of $\times$–boxes and Θ_1 has the same starting and ending points as α, or u is part of Θ_1.*
2. *Let p be the content difference of the rightmost boxes in the first two rows, then $|u| \leq p$.*

Proof. Let z be the rightmost box of the top row of λ/μ, then there must exist a proper tiling (Θ, ν) such that Θ_1 terminates at z because otherwise z would be a determined $\times$-box, which is against the assumption. Let $c(z) = c$ and y be the starting box of α, then $c(y) = c-k+1$. We claim that Θ_1 also starts at y. Suppose otherwise, the starting box of Θ_1 is to the southeast of y (along the same diagonal) because they have the same content and consequently the box right below y has to be $\times$ due to condition (ii). However, y is the topmost box of a column, which excludes the possibility that the box right below y could be $\times$ because of condition (i). As a result, Θ_1 must start at y, that is, both Θ_1 and α have the same starting and ending points. It follows that u is a row of $\times$-boxes in Θ as all boxes standing before y of the same row must be $\times$ by (ii).

Since all $\times$-boxes are undetermined, there exists a tiling Ω such that u as a sequence of $\bullet$-boxes is part of some Ω_i. We assert that $i = 1$. Suppose otherwise, that is, a k-ribbon Ω_j is located to the northeast of Ω_i and $j < i$. Therefore, the starting box of Ω_j is to the northeast of y, thus its content is strictly larger than $c - k + 1$, while the ending box of Ω_j has content at most c. This results in that $|\Omega_j| < k$, a contradiction. It follows that $i = 1$ and u is part of Ω_1. This completes the proof of (1).

We next show (2). Since all $\times$-boxes are undetermined and $u \neq \varnothing$, (1) guarantees the existence of a proper tiling (Φ, ν) such that u is part of Φ_1, and by Observation 3.4 Φ_1 ends at the rightmost box of the second row. Let z' be the rightmost box of the second row of λ/μ, then $c(z') = c(z) - p = c - p$, thus the starting box of Φ_1 has content $c - p - k + 1$. Since the leftmost box of u has content $c - k + 1 - |u|$ and it is to the northeast of the starting box of Φ_1, we must have $c - k + 1 - |u| \geq c - p - k + 1$, i.e., $|u| \leq p$, as desired. □

Proof of Theorem 1.4 (1). We argue it by induction on m. The base case $m = 1$ is resolved by Lemma 3.5. For $m \geq 2$, $n \geq 1$ and any connected skew diagram θ/ρ, let $r = (|\theta|-|\rho|-n)/k$. Suppose that $|\mathcal{O}_n^k(\theta, \rho)| = |\mathcal{E}_n^k(\theta, \rho)|$ for any $1 \leq r < m$ if $|\mathcal{D}_n^k(\theta, \rho)| \geq 2$. We will show that it is also true for all skew diagrams θ/ρ and $n \geq 1$ such that r equals m. We distinguish the cases that $u = \varnothing$, $1 \leq |u| < p$ or $|u| = p$ where p is defined in (2) of Lemma 3.6.

Case (1): $u = \varnothing$ and we will present a bijection $\sigma : \mathcal{D}_n^k(\lambda, \mu) \to \mathcal{D}_n^k(\lambda, \mu \,\dot{\cup}\, \alpha)$.

For any $(\Theta, \nu) \in \mathcal{D}_n^k(\lambda, \mu)$, let s be the union of $\times$–boxes right above Θ_1, then $\omega(\Theta_1, s) = (\alpha, s')$ where s' is the set of $\times$–boxes right below α. Let $\Theta - \Theta_1$ be obtained from Θ by removing Θ_1, then $(\Theta - \Theta_1, (\nu/s) \,\dot{\cup}\, s') \in \mathcal{D}_n^k(\lambda, \mu \,\dot{\cup}\, \alpha)$ and define

$$\sigma(\Theta, \nu) = (\Theta - \Theta_1, (\nu/s) \,\dot{\cup}\, s'). \tag{3.1}$$

Since ω is an involution, the map σ is also a bijection, thus obtaining $|\mathcal{D}_n^k(\lambda, \mu)| = |\mathcal{D}_n^k(\lambda, \mu \,\dot{\cup}\, \alpha)|$. An important point to note here is that $r(\Theta_1) = r(\alpha)$. This is true because the top row of

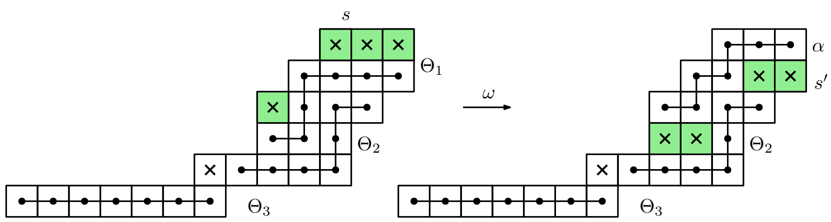

FIGURE 5. For the case $u = \varnothing$, a proper tiling (Θ, ν) and $\omega(\Theta_1, s)$ where $\sigma(\Theta, \nu)$ is obtained from the right by deleting α. The boxes of s and s' are highlighted in green.

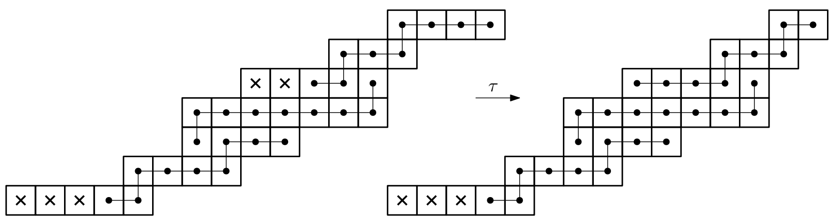

FIGURE 6. For the case $1 \le |u| < p$, an example of the bijection τ.

$\Theta_1 \dot{\cup} s$ is a row of $\times$-boxes if and only if the bottom row of $\alpha \dot{\cup} s'$ is a row of $\times$-boxes. In consequence, $|\mathcal{O}_n^k(\lambda, \mu)| = |\mathcal{E}_n^k(\lambda, \mu)|$ provided that $|\mathcal{O}_n^k(\lambda, \mu \dot{\cup} \alpha)| = |\mathcal{E}_n^k(\lambda, \mu \dot{\cup} \alpha)|$, which is true by induction hypothesis. See Figure 5 for an example of σ.

Case (2): $1 \le |u| < p$, we will reduce it to case (1) by a bijection

$$\tau : \mathcal{D}_n^k(\lambda, \mu) \to \mathcal{D}_{n-|u|}^k(\rho, \mu)$$

where ρ is obtained from λ by removing the rightmost $|u|$ boxes of its top row.

For $(\Theta, \nu) \in \mathcal{D}_n^k(\lambda, \mu)$, if Θ_1 ends at the rightmost box of the top row, then u must be a row of $\times$–boxes by Lemma 3.6. Let $\omega(\Theta_1, u) = (\beta, u')$, then β is the k-ribbon starting from the leftmost box of u and u' is the set of rightmost $\times$–boxes from the top row with $|u'| = |u|$. If Θ_1 terminates at the rightmost box of the second row, that is, the top row of λ/μ is occupied by $\times$–boxes, then remove the rightmost $|u|$ number of $\times$–boxes of the top row. Let $\beta = \Theta_1$.

Let η be the skew diagram such that η/μ is obtained from ν/μ by removing the rightmost $|u|$ boxes of the top row. Let $\Omega = \Theta - \Theta_1 + \beta$ be obtained from Θ by replacing Θ_1 by β, then define $\tau(\Theta, \nu) = (\Omega, \eta)$ and it is a bijection with the property $r(\Theta) = r(\Omega)$, implying that $|\mathcal{O}_n^k(\lambda, \mu)| = |\mathcal{O}_{n-|u|}^k(\rho, \mu)|$ and $|\mathcal{E}_n^k(\lambda, \mu)| = |\mathcal{E}_{n-|u|}^k(\rho, \mu)|$. See Figure 6 for an example of τ.

Note that the proper tiling (Ω, η) of ρ/μ belongs to case (1), that is, the k-ribbon that ends at the rightmost box of the top row starts from a leftmost box. It follows from the argument of case (1) that $|\mathcal{O}_{n-|u|}^k(\rho, \mu)| = |\mathcal{E}_{n-|u|}^k(\rho, \mu)|$, namely, $|\mathcal{O}_n^k(\lambda, \mu)| = |\mathcal{E}_n^k(\lambda, \mu)|$.

Case (3): $|u| = p$ and we shall prove that the first two rows overlap in exactly one column, that is, $p = \lambda_1 - \mu_1$. Let c be the content of the rightmost box of the first row of λ/μ. Suppose $p < \lambda_1 - \mu_1$, then there exists a proper tiling (Θ, ν) for which Θ_1 passes through all boxes of the top row, u is a sequence of $\times$–boxes and Θ_2 ends at the rightmost box of the second row, that is a box with content $c - p$. Therefore, the starting box of Θ_2 has content $c - p - k + 1$, which is also the content of the leftmost box of u. In consequence, the box right below the leftmost box of u must be $\times$ by (ii) as it stands before the starting box of Θ_2. However it is impossible due to (i). So we find that $|u| = p = \lambda_1 - \mu_1$.

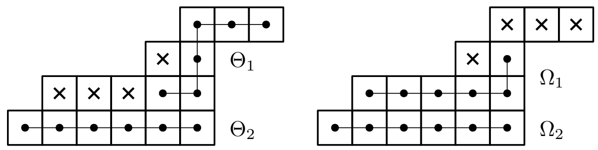

FIGURE 7. For the case $|u| = p$, two proper fillings with Θ_1 ending at the first row and Ω_1 terminating at the second row.

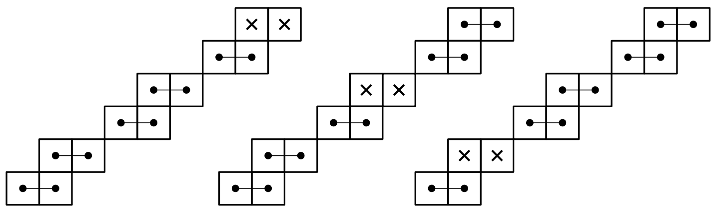

FIGURE 8. Three proper tilings of an edgewise disconnected skew diagram

This further leads to that the number of proper tilings of λ/μ with u as a row of $\times$–boxes equals the one with u as a row of $\bullet$–boxes and $r(\Theta_1)$ is changed by one, giving that $|\mathcal{O}_n^k(\lambda,\mu)| = |\mathcal{E}_n^k(\lambda,\mu)|$. Indeed, for any proper tiling with u as a row of $\times$-boxes, we interchange the boxes between u and the top row of Θ_1, obtaining a proper tiling with u as a row of $\bullet$-boxes. For example, see Figure 7. For all cases, we have shown that $|\mathcal{O}_n^k(\lambda,\mu)| = |\mathcal{E}_n^k(\lambda,\mu)|$ if $|\mathcal{D}_n^k(\lambda,\mu)| \geq 2$ inductively, which finishes the proof. $\square$

Remark 3.7. *It should be emphasized that Theorem 1.4 is not true for disconnected λ/μ. For instance, let $\lambda = (9,8,6,5,3,2)$ and $\mu = (7,6,4,3,1)$. Choose $k = 2$ and $|\nu/\mu| = 2$. There are three proper tilings of λ/μ as shown in Figure 8.*

3.1. **Proper tilings of a Young diagram.** We will turn to the special case $\mu = \varnothing$ and will build connections between Theorem 1.4 and the combinatorial meaning of $\mathrm{Pet}_k(\lambda,\varnothing)$ by Cheng *et al.* [4] in Section 4. For simplicity, set $\mathcal{D}_n^k(\lambda) = \mathcal{D}_n^k(\lambda,\varnothing)$. Let $\mathrm{core}_k(\lambda)$ denote the *k-core* obtained from λ by successively removing k-ribbons in a way that at each step what remains is also a Young diagram. It is independent of the order in which ribbons removed (cf. [15, p. 12, Ex. 8 (c)]). In this particular case, the cardinality of $\mathcal{D}_n^k(\lambda)$ is limited to precisely two alternatives 0 and 1.

Theorem 3.8. *For a partition λ, we have $|\mathcal{D}_n^k(\lambda)| \in \{0,1\}$. If $\lambda_1 < k$, then $|\mathcal{D}_n^k(\lambda)| = 1$ if and only if $\mathrm{core}_k(\lambda)$ has at most one row.*

Proof. First we show that if $\mathcal{D}_n^k(\lambda) \neq \varnothing$, then $|\mathcal{D}_n^k(\lambda)| = 1$ and $\mathrm{core}_k(\lambda)$ has at most one row.

If $\mathcal{D}_n^k(\lambda) \neq \varnothing$, that is, a proper k-ribbon tiling (Θ,ν) with $|\nu| = n$ exists, then $\nu = \mathrm{core}_k(\lambda)$ has to be a row (could be empty) by condition (i) and ν should start from the leftmost box of the top row of λ. That is, there is only one choice for ν, thus a unique choice for Θ as Θ is determined by ν, k and λ. As a result, $|\mathcal{D}_n^k(\lambda)| = 1$.

Second we establish that if $\lambda_1 < k$ and $\mathrm{core}_k(\lambda)$ has at most one row, then $|\mathcal{D}_n^k(\lambda)| = 1$. This is achieved by finding a proper k-ribbon tiling of λ. Choose Θ_1 as the k-ribbon from $\mathrm{NW}(\lambda/\mathrm{core}_k(\lambda))$ that ends at the northeasternmost box of $\lambda/\mathrm{core}_k(\lambda)$, then we assert that Θ_1 starts from the first column. Let c be the content of the northeasternmost box of $\lambda/\mathrm{core}_k(\lambda)$, then the starting box of Θ_1 has content $c - k + 1 = \lambda_1 - k \leq -1$. Since the leftmost box of

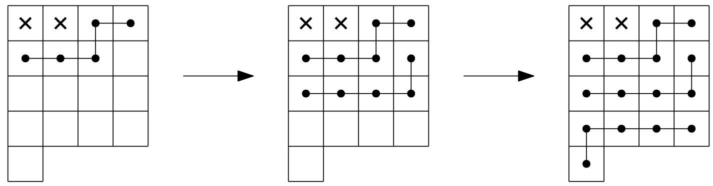

FIGURE 9. The existence of a unique proper tiling if $4 = \lambda_1 < k = 5$ and $\mathrm{core}_k(\lambda) = (2)$ is a row.

the second row has content -1 and Θ_1 is part of $\mathrm{NW}(\lambda/\,\mathrm{core}_k(\lambda))$, we see that Θ_1 must start with the first column of λ.

Let d be the content of the northeasternmost box of $\lambda/(\mathrm{core}_k(\lambda)\,\dot{\cup}\,\Theta_1)$, then $d < c$ because if $d = c$, then a box of content $c+1$ is contained in λ, contradicting that c is the largest content. Define Θ_2 to be the k-ribbon from $\mathrm{NW}(\lambda/(\mathrm{core}_k(\lambda)\,\dot{\cup}\,\Theta_1))$ that terminates at the box of content d, that is also the northeasternmost box of $\lambda/(\mathrm{core}_k(\lambda)\,\dot{\cup}\,\Theta_1)$. Then Θ_2 starts from a box of content $d-k+1 < c-k+1$. Since the starting box of Θ_1 has content $c-k+1$ and Θ_2 is part of $\mathrm{NW}(\lambda/(\mathrm{core}_k(\lambda)\,\dot{\cup}\,\Theta_1))$, the starting box of Θ_2 also stands at the first column.

Continue this process until we obtain Θ_m for which $(\Theta, \mathrm{core}_k(\lambda))$ with $\Theta = (\Theta_1, \dots, \Theta_m)$ is a proper tiling of λ as all Θ_i's start from the first column of λ. See Figure 9 for an example. This finishes the proof. □

4. THE PROOF OF THEOREM 1.4

The purpose of this section is to prove the second part of Theorem 1.4, that is (2), characterizing k-Petrie numbers $\mathrm{Pet}_k(\lambda, \mu)$ in terms of proper tilings.

First we write the k-Petrie numbers in terms of skew Schur functions via λ-ring notations (see Lemma 4.1). Second we apply the plethystic Murnaghan-Nakayama rule to express the k-Petrie numbers as a sum of signed proper tilings. Finally prove (1.4) by applying (1) of Theorem 1.4.

Lemma 4.1. *Let $\hat{\omega} = 1 + \omega + \cdots + \omega^{k-1}$ with $\omega = e^{\frac{2\pi i}{k}}$. Then*

$$\mathrm{Pet}_k(\lambda, \mu) = s^{\lambda/\mu}(1 - \hat{\omega}). \tag{4.1}$$

Proof. The Jacobi-Trudi identity (2.11) and (2.7) yield

$$s^{\lambda/\mu}(1-\hat{\omega}) = \det_{1 \le i,j \le \ell(\lambda)} \left(\sigma^{\lambda_i - \mu_j - i + j}(1 - \hat{\omega}) \right). \tag{4.2}$$

We next simplify $\sigma^r(1-\hat{\omega})$ by using (2.4), (2.2) and (2.9) as follows.

$$\begin{aligned} \sum_{r=0}^{\infty} \sigma^r(1-\hat{\omega})\, t^r &= \sigma_t(1-\hat{\omega}) = \sigma_t\left(-\sum_{i=1}^{k-1} \omega^i\right) \\ &= \sigma_t\left(\sum_{i=1}^{k-1}\omega^i\right)^{-1} = H(t)^{-1}\Big|_{\substack{x_i = \omega^i,\, 1 \le i < k \\ x_i = 0,\, i \ge k}} \\ &= \prod_{i=1}^{k-1}(1-\omega^i t) = \frac{1-t^k}{1-t} = 1 + t + \cdots + t^{k-1}. \end{aligned}$$

which further results in $\sigma^r(1-\hat{\omega}) = 1$ for $0 \le r < k$ and $\sigma^r(1-\hat{\omega}) = 0$ otherwise. Here the second to last equality comes from the last equality of (1.1). Plugging it to (4.2), together with the determinantal formula of $\mathrm{Pet}_k(\lambda,\mu)$ in Theorem 1.1, gives (4.1), which completes the proof. □

This lemma combined with a multiparameter Murnaghan-Nakayama rule in [10] provides another proof of (1.3); see Remark 4.2. Therefore, it is advantageous to write the Petrie symmetric functions in the λ-ring notations.

Remark 4.2. *The equality* (1.1) *is equivalent to*

$$G(k,m)(x) = \sigma^m((1-\hat{\omega})x), \tag{4.3}$$

by the following argument on the formal power series

$$\begin{aligned}\sum_{m=0}^{\infty} \sigma^m((1-\hat{\omega})x)\, t^m &= \sigma_t((1-\hat{\omega})x) = \sigma_t\left(x - \sum_{i=0}^{k-1} \omega^i x\right) \\ &= \sigma_t(x)\sigma_t\left(\sum_{i=0}^{k-1}\omega^i x\right)^{-1} = H(t)\prod_{i=0}^{k-1} H(\omega^i t)^{-1} = \sum_{m=0}^{\infty} G(k,m)(x)t^m.\end{aligned} \tag{4.4}$$

These equalities are true because of (2.4)*,* (2.2) *and* (2.9)*. Let us recall an identity from* [10, Corollary 3.2] *$(q=t=0)$:*

$$\sigma^m((\mathbf{a}-\mathbf{b})x)s_\mu(x) = \sum_{\lambda \vdash m+|\mu|} s^{\lambda/\mu}(\mathbf{a}-\mathbf{b})s_\lambda(x), \tag{4.5}$$

where $\mathbf{a}$ *and* $\mathbf{b}$ *are sequences of parameters. Take*

$$\mathbf{a} = (1,0,\ldots) \quad \textit{and} \quad \mathbf{b} = (1,\omega,\ldots,\omega^{k-1},0\ldots),$$

so that $(\mathbf{a}-\mathbf{b})x = (1-\hat{\omega})x$*, then we conclude* (1.3) *by combining* (4.3)*,* (4.1) *and* (4.5)*.*

Some properties of $\mathrm{Pet}_k(\lambda,\mu)$ are to be discussed, which reduces the problem to finding combinatorial meanings for $\mathrm{Pet}_k(\lambda,\mu)$ when λ/μ is connected with row-size bounded by k.

Lemma 4.3. *For partitions μ,λ such that $\mu \subseteq \lambda$, we have*
(1) if there exists i such that $\lambda_i - \mu_i \ge k$, then $\mathrm{Pet}_k(\lambda,\mu) = 0$*;*
(2) if λ/μ has connected components ρ_i/ξ_i for $1 \le i \le m$, then $\mathrm{Pet}_k(\lambda,\mu) = \prod_{i=1}^m \mathrm{Pet}_k(\rho_i,\xi_i)$*.*

Proof. Define two matrices $A = (a_{ij}) = (\lambda_i - \mu_j - i + j)$ and $M = (m_{ij}) = (\chi(0 \le a_{ij} < k))$. As one can see, the entries in A along each row (or column) are strictly increasing from left to right (decreasing from top to bottom). If $\lambda_i - \mu_i \ge k$, then all entries of A to the northeast of the (i,i)-entry are larger than or equal to k, i.e., $m_{kl} = 0$ for all $k \le i$ and $l \ge i$. Applying the Laplace expansion to the last column of M produces $\det(M) = 0$ inductively, which gives (1).

For (2), let j be the smallest integer for which $\mu_j \ge \lambda_{j+1}$, that is, λ/μ is connected till the j-th row. In this case $a_{j+1,j} = \lambda_{j+1} - \mu_j - 1 < 0$, which yields that $a_{kl} < 0$ for all $k \ge j+1$ and $l \le j$. Equivalently $m_{kl} = 0$ for all $k \ge j+1$ and $l \le j$. Applying the Laplace expansion to the first column of M inductively leads to $\mathrm{Pet}_k(\lambda,\mu) = \mathrm{Pet}_k(\rho_1,\xi_1)\mathrm{Pet}_k(\lambda/\rho_1,\mu/\xi_1)$. Repeating this process gives (2). □

We are ready to prove (1.4).

Proof of Theorem 1.4 (2). By (2.10) and (2.12), we have

$$s^{\lambda/\mu}(1-\hat{\omega}) = \sum_{\mu\subseteq\nu\subseteq\lambda} s^{\lambda/\nu}(-\hat{\omega})s^{\nu/\mu}(1) = \sum_{\nu}(-1)^{|\lambda/\nu|}s^{\lambda'/\nu'}(\hat{\omega}) \tag{4.6}$$

summed over all partitions $\mu \subseteq \nu \subseteq \lambda$ such that ν/μ is a horizontal strip. This is true because $s^{\nu/\mu}(1) = s_{\nu/\mu}(1) = 1$ if and only if ν/μ is a horizontal strip.

The plethysm $p_k \circ h_n$ [15, Page 135] is defined by $p_k \circ h_n = h_n(x_1^k, x_2^k, \cdots)$. Consider the generating function of $p_k \circ h_n$, we find

$$\sum_{n=0}^{\infty} p_k \circ h_n z^{kn} = \sum_{n=0}^{\infty} h_n(x_1^k, x_2^k, \cdots)z^{kn} = \prod_{i\geq 1}\frac{1}{1-x_i^k z^k} = \prod_{i\geq 1}\prod_{j=0}^{k-1}\frac{1}{1-\omega^j x_i z} = \sum_{n=0}^{\infty}\sigma^n(\hat{\omega}x)z^{kn}.$$

This proves $p_k \circ h_n = \sigma^n(\hat{\omega}x)$. Together with (4.5) for $m = kn$,

$$\mathbf{a} = (1, \omega, \ldots, \omega^{k-1}, 0, \ldots) \text{ and } \mathbf{b} = (0, 0, \ldots),$$

we arrive at

$$p_k \circ h_n(x)s_\nu(x) = \sigma^n(\hat{\omega}x)s_\nu(x) = \sum_{\lambda\vdash kn+|\mu|} s^{\lambda/\nu}(\hat{\omega})s_\lambda(x). \tag{4.7}$$

On the other hand, it follows from [17, (2)] (see also [3, 5, 7]) that

$$p_k \circ h_n(x)s_\nu(x) = \sum_{\lambda\vdash kn+|\nu|} \mathrm{sgn}_k(\lambda,\nu)s_\lambda(x) \tag{4.8}$$

summed over partitions λ with a (unique) k-ribbon tiling Θ of λ/ν satisfying condition (ii$'$). In this case, we call λ/ν *horizontal k-tileable* and

$$\mathrm{sgn}_k(\lambda,\nu) := \prod_{i\geq 1}(-1)^{ht(\Theta_i)}.$$

Combining (4.7) and (4.8) gives that

$$s^{\lambda/\nu}(\hat{\omega}) = \begin{cases} \mathrm{sgn}_k(\lambda,\nu) & \text{if } \lambda/\nu \text{ is horizontal } k\text{-tileable;} \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

If ξ is a ribbon, then $r(\xi) + ht(\xi') = |\xi|$, by which (4.9) is equivalent to

$$s^{\lambda'/\nu'}(\hat{\omega}) = \begin{cases} (-1)^{|\lambda/\nu|}\prod_{i\geq 1}(-1)^{r(\Theta_i)} & \text{if } \lambda/\nu \text{ is vertical } k\text{-tileable;} \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Here λ/ν is *vertical k-tileable* if there exists a (unique) k-ribbon tiling Θ' of λ/ν satisfying condition (ii). Substituting (4.10) to (4.6) yields that $s^{\lambda/\mu}(1-\hat{\omega}) = 0$ unless λ/μ has at least one proper k-ribbon tiling, i.e., $|\mathcal{D}_n^k(\lambda,\mu)| \geq 1$ for some n. In this case,

$$s^{\lambda/\mu}(1-\hat{\omega}) = \sum_{n\geq 0}\sum_{(\Theta,\nu)\in\mathcal{D}_n^k}\prod_{i\geq 1}(-1)^{r(\Theta_i)} \tag{4.11}$$

where $\mathcal{D}_n^k = \mathcal{D}_n^k(\lambda,\mu)$. There are an even number of 1's or -1's appearing in the second summation of (4.11) as long as $|\mathcal{D}_n^k(\lambda,\mu)| \neq 1$ by (1) of Theorem 1.4. This, together with Lemma 4.1, suggests that $\mathrm{Pet}_k(\lambda,\mu) = s^{\lambda/\mu}(1-\hat{\omega}) = 0$ unless $|\mathcal{D}_n^k(\lambda,\mu)| = 1$, by which and Lemma 4.3 (1) we conclude that (1.4) is true for connected λ/μ with row-size less than k.

One can easily extend (1.4) to disconnected skew diagrams by Lemma 4.3 (2), which finishes the proof. □

Specializing $\mu = \varnothing$ in Theorem 1.4 and Combining Theorem 3.8, we obtain the result below firstly in [4, Theorem 1.3].

Corollary 4.4. $\mathrm{Pet}_k(\lambda) = s^{\lambda}(1-\hat{\omega}) = 0$ *unless* $\lambda_1 < k$ *and* $\mathrm{core}_k(\lambda)$ *has at most one row. In this case, if* $\gamma^{(1)}, \dots, \gamma^{(\ell)}$ *is any sequence of partitions such that*

$$\mathrm{core}_k(\lambda) = \gamma^{(0)} \subseteq \gamma^{(1)} \subseteq \cdots \subset \gamma^{(\ell-1)} \subseteq \gamma^{(\ell)} = \lambda$$

and $\gamma^{(j)}/\gamma^{(j-1)}$ *is a* k*-ribbon for all* $1 \le j \le \ell$*. Then* $\mathrm{Pet}_k(\lambda) = 1$ *if* $\mathrm{core}_k(\lambda) = \lambda$*; otherwise*

$$\mathrm{Pet}_k(\lambda) = \prod_{j=1}^{\ell} (-1)^{r(\gamma^{(j)}/\gamma^{(j-1)})}. \tag{4.12}$$

Proof. It follows from Theorem 3.8, Theorem 1.4 and that the RHS of (4.12) is independent of different ribbon decompositions of λ (see for instance [1, §6], [12, Proposition 3.3.1] and [16, Lemma 4.1]). □

With the help of Theorem 1.4, it is easier to derive two specializations, respectively of skew Schur functions and of the Pieri-like rule for the Petrie symmetric functions.

4.1. **A specialization of Schur functions.** Equation (4.9) presents a combinatorial interpretation for $s_{\lambda/\mu}(1, \omega, \cdots, \omega^{k-1})$; see also [15, p. 91, Ex. 24(b)]. Now we shall exhibit an analogous description for $s_{\lambda/\mu}(\omega, \omega^2 \cdots, \omega^{k-1})$ as a direct consequence of Theorem 1.4.

Corollary 4.5. *Let* μ, λ *be two partitions such that* $\mu \subseteq \lambda$ *and* $\omega = e^{\frac{2\pi i}{k}}$*. Then we have*

$$s_{\lambda/\mu}(\omega, \omega^2 \cdots, \omega^{k-1}) = (-1)^{|\nu/\mu|} \prod_{i=1}^{m} (-1)^{ht(\Theta_i)} \tag{4.13}$$

if λ/μ *has only one dual proper tiling* $(\Theta_1, \cdots, \Theta_m, \nu)$ *and every column of* λ/μ *has size less than* k*; otherwise* $s_{\lambda/\mu}(\omega, \omega^2 \cdots, \omega^{k-1}) = 0$.

Proof. By the duality rule (2.10), we have

$$s_{\lambda/\mu}(\omega, \omega^2 \cdots, \omega^{k-1}) = s^{\lambda/\mu}(\hat{\omega} - 1) = (-1)^{|\lambda/\mu|} s^{\lambda'/\mu'}(1-\hat{\omega}).$$

Then (4.13) holds by taking conjugation on (1.4). □

4.2. **A specialization of (1.3).** As we mentioned in the introduction, one of the motivations to study Petrie symmetric functions is the Liu-Polo conjecture [9], which is now a corollary of Theorem 1.1. Here we use f to denote $f(x)$.

Corollary 4.6. [4, 9] *For* $k \in \mathbb{N}^+ \backslash \{1\}$*, we have*

$$G(k,k) = \sum_{\substack{\lambda \vdash k \\ \lambda \trianglelefteq (k-1,1)}} m_\lambda = \sum_{i=0}^{k-2} (-1)^i s_{(k-1-i, 1^{i+1})},$$

$$G(k, 2k-1) = \sum_{\substack{\lambda \vdash 2k-1 \\ \lambda \trianglelefteq (k-1,k-1,1)}} m_\lambda = \sum_{i=0}^{k-2} (-1)^i s_{(k-1,k-1-i,1^{i+1})},$$

where $\trianglelefteq$ is the dominance order, that is, $\mu \trianglelefteq \lambda$ if $|\lambda| = |\mu|$ and $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all $i \geq 1$.

We consider an extension of Corollary 4.6, that is the case of (1.3) when μ is a row partition.

Corollary 4.7. *Let $k \in \mathbb{N}^+\backslash\{1\}$ and $r \in \mathbb{N}$. Then*

$$G(k,k)h_r = \sum_{i=1}^{r} s_{(r+k-i,i)} + \sum_{i=0}^{k-r-2} (-1)^i s_{(k-i-1,r+1,1^i)} + \sum_{i=1}^{r} (-1)^{k-i+1} s_{(r,i,1^{k-i})}, \quad (4.14)$$

$$G(k,2k-1)h_r = \sum_{i=1}^{\min(k-1,r)} \sum_{j=1}^{i} (-1)^{k-j+1} s_{(r+k-1-i,i,j,1^{k-j})} \quad (4.15)$$

$$+ \sum_{i=0}^{\min(k-2,r)} \sum_{j=0}^{k-2-i} (-1)^j s_{(r+k-1-i,k-1-j,i+1,1^j)}.$$

Proof. By definition, $h_r = s_{(r)}$. Theorem 1.4 tells us that the coefficient of s_λ on the RHS of (4.14) is non-zero if and only if λ is obtained from (r) by adding a horizontal strip or a ribbon of size k and each row-size of $\lambda/(r)$ is less than k.

If $\lambda/(r)$ is a horizontal strip of size k, then $\lambda/(r)$ must have exactly two rows, that is, $\lambda = (r+k-i,i)$ for $1 \leq i \leq r$ with coefficient 1 by (1.4). Note that $i \neq 0$ because every row of $\lambda/(r)$ has less than k boxes. This contributes the first term of (4.14).

If $\lambda/(r)$ is a k-ribbon, then $r \leq k-1$. If $\lambda/(r)$ is a hook, that is a ribbon and a partition, we must have $\lambda = (r,i,1^{k-i})$ for $1 \leq i \leq r$ with coefficient $(-1)^{k+1-i}$; otherwise $\lambda/(r)$ has a box from the first row, that is, $\lambda = (k-i-1,r+1,1^i)$ for $0 \leq i \leq k-r-2$ with coefficient $(-1)^i$. These correspond to the second and the third terms of (4.14), as wished.

For the formula of $G(k,2k-1)h_r$, since each part of $\lambda/(r)$ is less than k, the partition λ can only be obtained by adding firstly a horizontal strip of size $k-1$ and then a k-ribbon, namely $\nu = (r+k-1-i,i)$ for $0 \leq i \leq \min(k-1,r)$. In addition, the ending box of the unique k-ribbon is not at the first row. If λ/ν is a hook and ν has two rows, then we have $\lambda = (r+k-1-i,i,j,1^{k-j})$ for $1 \leq i \leq k-1$, and $i \leq r$ with coefficient $(-1)^{k-j+1}$, which yields the first term of (4.15); otherwise if λ/ν is a hook and $\nu = (r+k-1)$ is a row, or λ/ν is not a hook, then $\lambda = (r+k-1-i,k-1-j,i+1,1^j)$ where $0 \leq i \leq k-2$ and $i \leq r$ with coefficient $(-1)^j$. This corresponds to the second term of (4.15) and we complete the proof. □

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References

[1] P. Alexandersson, S. Pfannerer, M. Rubey and J. Uhlin, *Skew characters and cyclic sieving*, Forum Math. Sigma 9 (2021), Paper No. e41, 31pp. 15

[2] A. Bazeniar, M. Ahmia and H. Belbachir, *Connection between bi^{s}nomial coefficients and their analogs and symmetric functions*, Turkish J. Math. 42 (2018), no. 3, 807-818. 1

[3] Y. Cao, N. Jing and N. Liu, *Plethystic Murnaghan-Nakayama rule via vertex operators*, preprint, arXiv:2212.08412. 2, 14

[4] Y. Cheng, M. Chou, S. Eu, T. Fu and J. Yao, *On signed multiplicities of Schur expansions surrounding Petrie symmetric functions*, Proc. Amer. Math. Soc. 151 (2023), no. 5, 1839-1854. 3, 11, 15

[5] J. Désarménien, B. Leclerc and J. Y. Thibon, *Hall-Littlewood functions and Kostka-Foulkes polynomials in representation theory*. Sém. Lothar. Combin. 32 (1994), Art. B32c, approx. 38 pp. 2, 14

[6] S. Doty, G. Walker, *Modular symmetric functions and irreducible modular representations of general linear groups*, J. Pure Appl. Algebra 82 (1992), no. 1, 1-26. 1, 3

[7] A. Evseev, R. Paget and M. Wildon, *Character deflations and a generalization of the Murnaghan-Nakayama rule*. J. Group Theory 17 (2014), no. 6, 1035-1070. 2, 14

[8] H. Fu, Z. Mei, *Truncated homogeneous symmetric functions*, Linear Multilinear Algebra 70 (2022), no. 3, 438-448. 1

[9] D. Grinberg, *Petrie symmetric functions*, Algebr. Comb. 5 (2022), no. 5, 947-1013. 1, 15

[10] N. Jing, N. Liu *A multiparametric Murnaghan-Nakayama rule for Macdonald polynomials*, J. Combin. Theory Ser. A 207 (2024), Paper No. 10592032 (34pp). 13

[11] D. Knutson, *λ-rings and the representation theory of the symmetric group*, Springer, 1973. 3, 5

[12] M. A. A. van Leeuwen, *Edge sequences, ribbon tableaux, and an action of affine permutations*, European J. Combin. 20 (1999), 179-195. 15

[13] L. Liu, *On the cohomology of line bundles over certain flag schemes II*, J. Combin. Theory Ser. A 182 (2021), Paper No. 105448. 1

[14] L. Liu, P. Polo, *On the cohomology of line bundles over certain flag schemes II*, J. Combin. Theory Ser. A 178 (2021), Paper No. 105352, 11 pp. 1

[15] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd Edition, Clarendon Press, Oxford, 1995. 2, 3, 5, 6, 11, 14, 15

[16] I. Pak, *Ribbon tile invariants*, Trans. Amer. Math. Soc. 352 (2000), 5525-5561. 15

[17] M. Wildon, *A combinatorial proof of a plethystic Murnaghan-Nakayama rule*. SIAM J. Discrete Math. 30 (2016), no. 3, 1526-1533. 2, 14

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