Equality of thickened ribbon Schur functions

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XMU–XMUM Joint Seminar
April 29, 2024

1. Symmetric functions in commuting variables

Symmetric functions

Let $X = (x_1, x_2, \ldots)$ be a set of variables, a homogeneous symmetric function of degree n over a commutative ring R (with identity) is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha$$

where (i) α ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n, (ii) $c_\alpha \in R$, (iii) x^α stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, and (iv) $f(x_{w(1)}, x_{w(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation w of the positive integers P. For example, $n = 2$,

$$m_{11}(x) = \sum_{i<j} x_i x_j = x_1x_2 + x_1x_3 + x_2x_3 + \cdots$$

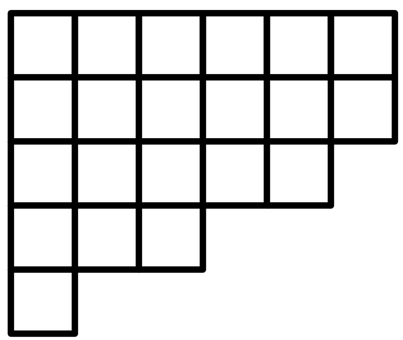
$$m_2(x) = \sum_{i} x_i^2 = x_1^2 + x_2^2 + x_3^2 + \cdots$$

are symmetric functions of degree 2.

Symmetric functions

Let Λ^n denote the vector space consisting of symmetric functions of homogeneous degree n, then $\dim(\Lambda^n) = \#$ partitions of n.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a weakly decreasing sequence of positive integers (i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$) such that the sum of λ_i's equals n.



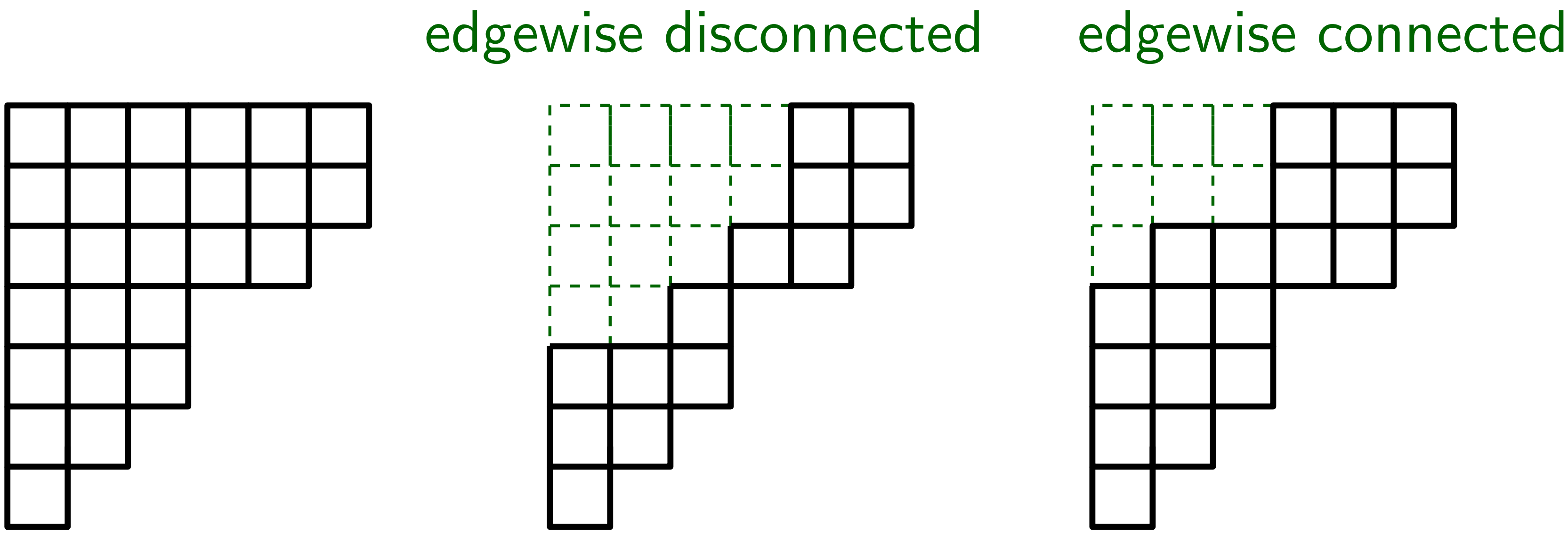
$$\lambda = (6, 6, 5, 3, 1)$$

The vector space bases for Λ^n: monomial symmetric functions m_λ, elementary symmetric functions e_λ, complete homogeneous symmetric functions h_λ, power–sum symmetric functions p_λ and Schur functions s_λ.

2. Young tableaux and skew Schur functions

Semistandard Young Tableaux (SSYT)

A skew partition λ/μ of *n* is the set-theoretic difference λ/μ of the Young diagrams, which consists of *n* boxes. For a skew diagram, if there are two boxes that do not share an edge, then we call this diagram edgewise disconnected; otherwise edgewise connected.



$\lambda = (6, 6, 5, 3, 3, 2, 1)$ $\quad \mu = (4, 4, 3, 2)$ $\quad \mu = (3, 3, 1)$

Semistandard Young Tableaux (SSYT)

A Semistandard Young tableau (SSYT) of shape λ/μ is a Young diagram whose boxes have been filled with positive integers such that all entries along each row from left to right are weakly increasing, while all entries along each column from top to bottom are strictly increasing.

strictly increasing

1	1	3	4	4	5
2	3	4	5	6	8
4	6	6	7	9	
5	7	7			
7	8	8			
9	9				
10					

weakly increasing

strictly increasing

			1	2	2
			2	4	4
	1	3	4	5	
1	5	6			
3	7	7			
5	8				
8					

weakly increasing

Semistandard Young Tableaux (SSYT)

The skew Schur function $s_{\lambda/\mu}(x)$ of shape λ/μ in the infinitely many variables $x = (x_1, x_2, \ldots,)$ is the formal power series

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

summed over all SSYTs of shape λ/μ. In particular if $\mu = \emptyset$, we call $s_\lambda(x)$ the Schur function of shape λ.

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$$

$$s_{21}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3.$$

$$\begin{ytableau} \none & 1 \\ 1 & 2 \end{ytableau} \quad \begin{ytableau} \none & 1 \\ 2 & 2 \end{ytableau} \quad \begin{ytableau} \none & 1 \\ 1 & 3 \end{ytableau} \quad \begin{ytableau} \none & 1 \\ 3 & 3 \end{ytableau} \quad \begin{ytableau} \none & 2 \\ 2 & 3 \end{ytableau} \quad \begin{ytableau} \none & 2 \\ 3 & 3 \end{ytableau} \quad \begin{ytableau} \none & 1 \\ 2 & 3 \end{ytableau} \quad \begin{ytableau} \none & 2 \\ 1 & 3 \end{ytableau}$$

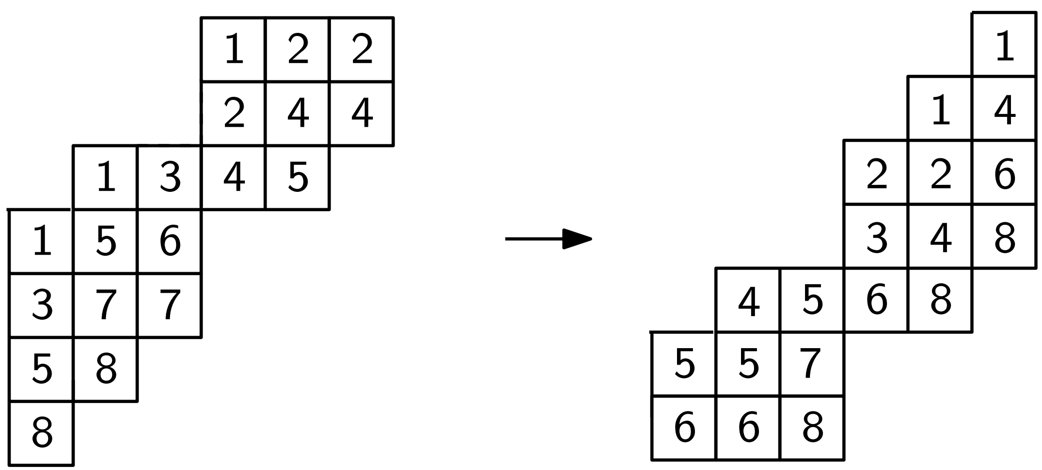
$$s_{(22)/(1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3.$$

Semistandard Young Tableaux (SSYT)

For every skew diagram $\mathcal{D}$, let $\mathcal{D}^*$ be the 180 antipodal rotation of $\mathcal{D}$, then we have

$$s_{\mathcal{D}} = s_{\mathcal{D}^*},$$

which follows by a simple bijection between two SSYT, that is, $i \mapsto N + 1 - i$ for every entry i where N is the largest entry. This shows us that two different skew diagrams could correspond to the same Schur function.

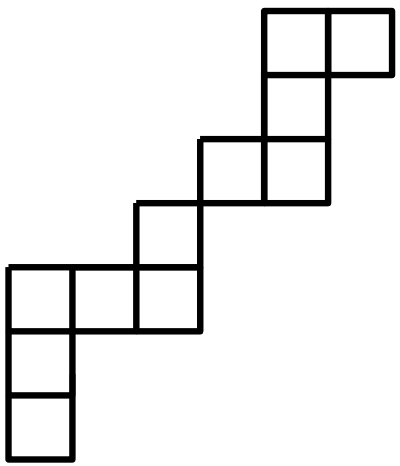


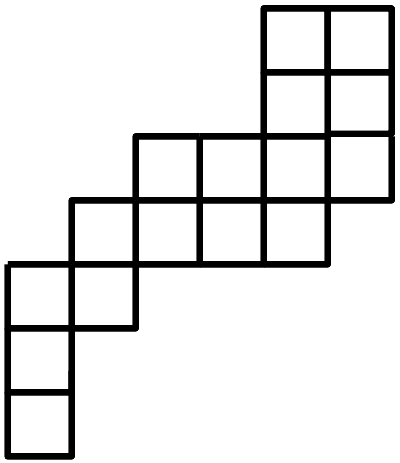
SSYT($\mathcal{D}$) SSYT($\mathcal{D}^*$)

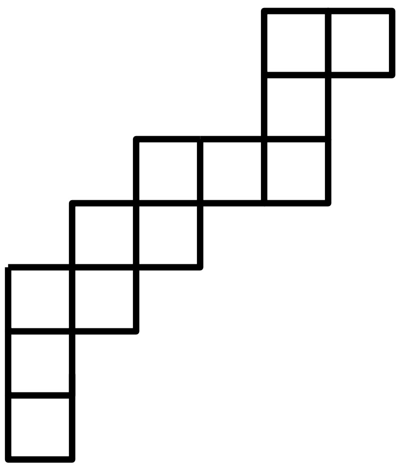
3. Ribbons and thickened ribbons

Ribbons and thickened ribbons

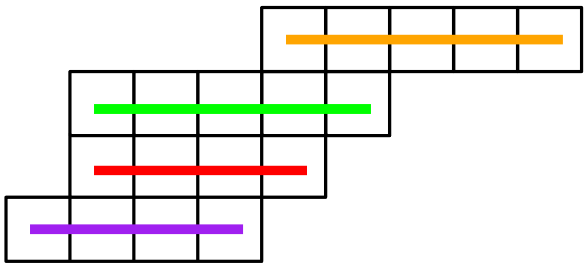
A ribbon is an edgewise connected skew shape without 2×2 blocks of boxes. A thickened ribbon is an edgewise connected skew shape without 3×3 block of boxes.



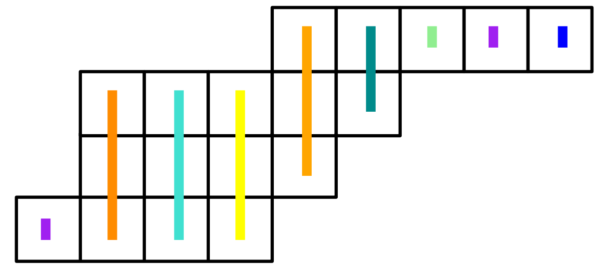




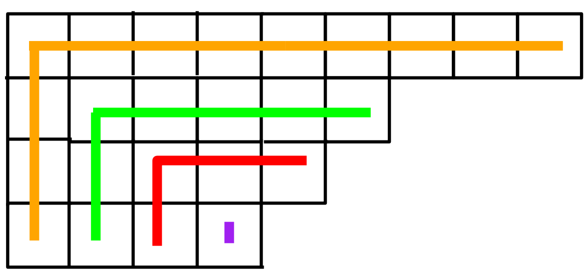
Skew Schur function determinants



Jacobi-Trudi determinant

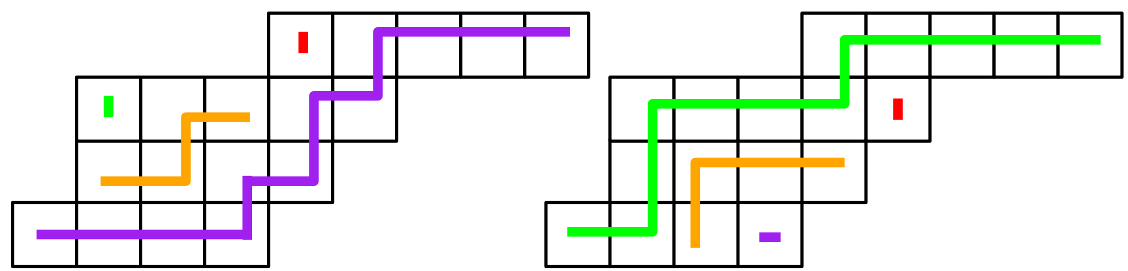


Dual Jacobi-Trudi determinant



Giambelli determinant

(only standard shape)



Lascoux and Pragacz determinant

choose the outer/inner ribbons

Ref. Hamel and Goulden (1995) unified these determinants.

Chen, Yan and Yang (2005) introduced cutting strips.

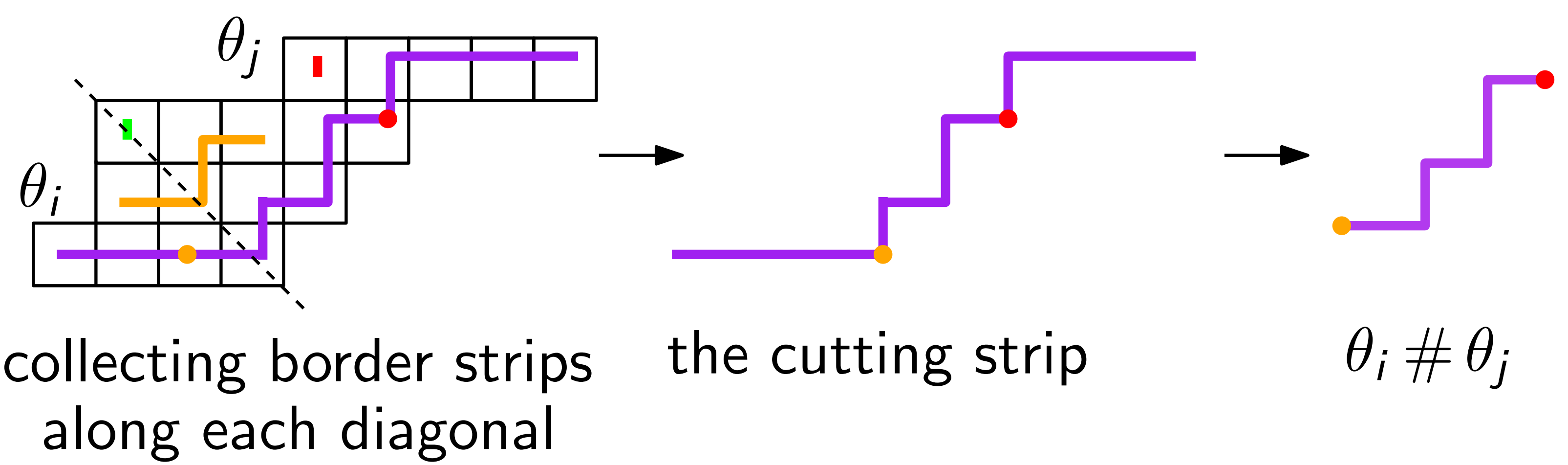
Skew Schur function determinants

Theorem (Hamel and Goulden, 1995). If the skew diagram of λ/μ is edgewise connected. Then, for any outside decomposition $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ of skew shape λ/μ, we have

$$s_{\lambda/\mu}(x) = \det[s_{\theta_i \# \theta_j}(x)]_{i,j=1}^{k},$$

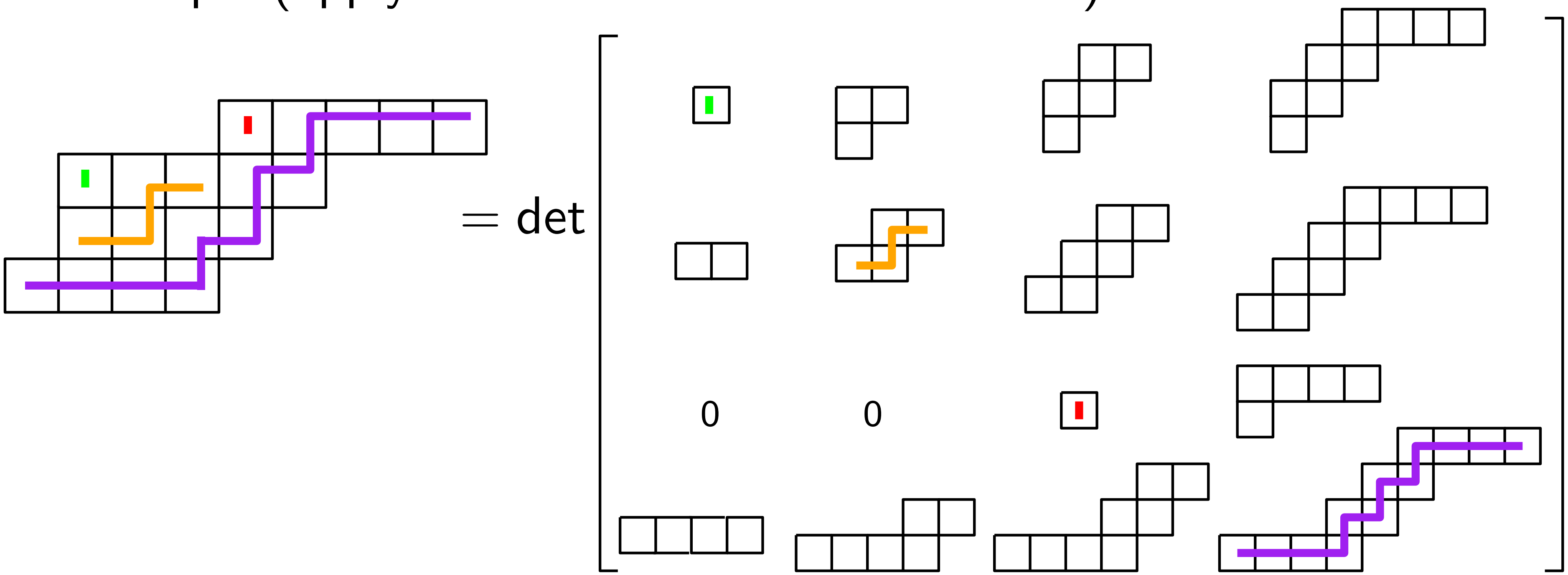
where $s_{\emptyset}(x) = 1$ and $s_{\theta_i \# \theta_j}(x) = 0$ if $\theta_i \# \theta_j$ is undefined.

Chen, Yan and Yang (2005) largely simplified the definition of $\theta_i \# \theta_j$ by introducing the concept of cutting strips.



Skew Schur function determinants

Example (apply Hamel and Goulden Theorem):



Here a skew shape $\mathcal{D}$ refers to the corresponding skew Schur function $s_{\mathcal{D}}(x)$. By the Hamel–Goulden Theorem, we always choose the minimal number of border strips in an outside decomposition, so that the order of the determinant is minimized.

Skew Schur function determinants

Theorem (Jin 2018). If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_k)$ of skew shape λ/μ, let $r = \sum_{i=1}^{k} |\Theta_i| - |\lambda/\mu|$, then

$$s_1(x)^r s_{\lambda/\mu}(x) = \det[s_{\Theta_i \,\#\, \Theta_j}(x)]_{i,j=1}^{k},$$

where $s_1(x)^r = (x_1 + x_2 + \cdots)^r$.

Remark 1: All Θ_i's are thickened border strips. If all Θ_i's are disjoint strips, then we retrieve HG Theorem.

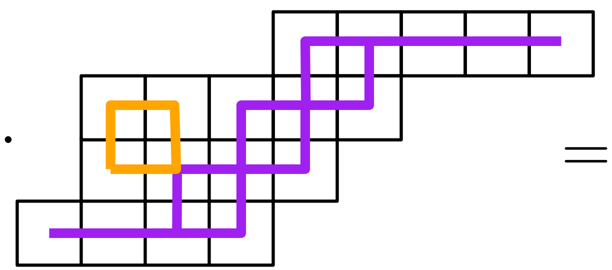
Remark 2: Through outside nested decomposition, we can further reduce the order of the skew Schur determinants.

Remark 3: We allow thickened strips to have common boxes.

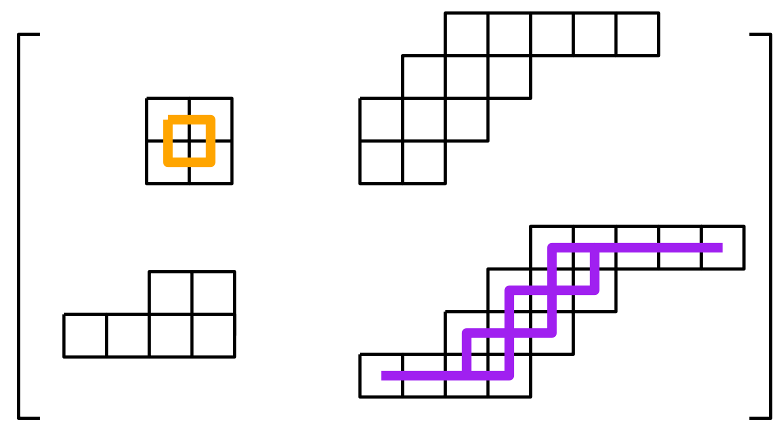
Skew Schur function determinants

Example (apply the J. Theorem):

$(\sum_{i=1}^{\infty} x_i)\cdot$

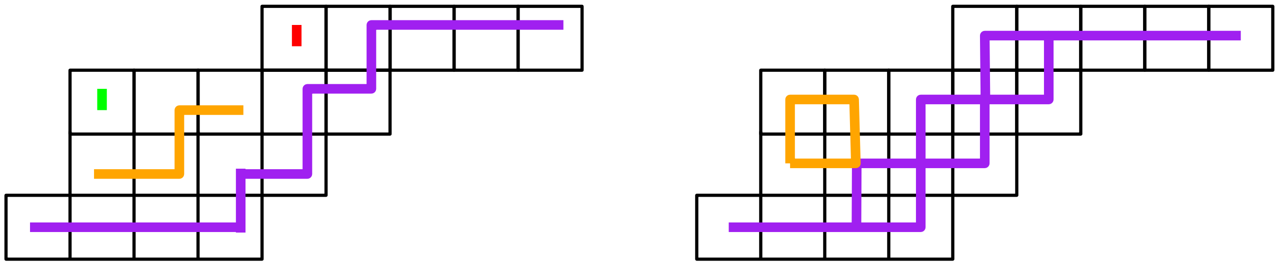


$= \det$



Here a skew shape $\mathcal{D}$ refers to the corresponding skew Schur function $s_{\mathcal{D}}(x)$. By the previous theorem, we obtain a simplified determinant of order 2. Each thickened border strip in contrast with border strips may contain 2×2 blocks of boxes. Subsequently we will see that each thickened border strip is not difficult to count.

Skew Schur function determinants



Our innovative part: we allow the lattice paths to be intersecting and control the intersecting parts within certain range so that they still cancel each other under the involution. In other words, we took non-intersecting approach to resolve certain intersecting lattice paths problem.

Subsequently, Kim and Yoo (2021) extended our results to a determinantal formula for the product of Schur functions satisfying certain conditions by applying the Bazin identity on determinants.

4. Equality on skew Schur functions

Motivations

Since both Schur functions $\{s_\lambda\}$ and monomial symmetric functions $\{m_\lambda\}$ for all paritithions λ of n are two bases of the space of homogeneous symmetric functions of degree n, the skew Schur function $s_{\lambda/\mu}$ has two expansions:

$$s_{\lambda/\mu}(x) = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_\nu$$

$$s_{\lambda/\mu}(x) = \sum_{\nu} K^{\nu}_{\lambda/\mu} m_\nu$$

where $c^{\lambda}_{\mu,\nu}$ is called the Littlewood–Richardson (LR) coefficient and $K^{\nu}_{\lambda/\mu}$ the Kostka number. Both are nonnegative. The LR coefficient has combinatorial interpretations in the Young tableaux model, the honeycomb model and the puzzle model, and appears in the representation theory and algebraic geometry, etc.
The Kostka number counts the number of SSYTs of a given skew shape and a filling type.

Motivations

Because of these rich connections, it becomes even more important to investigate properties of these nonnegative coefficients such as the conditions on the partitions λ, μ, ν for which $c^{\lambda}_{\mu\nu}$ is positive, and the typical Kostka number for uniformly random partitions λ, μ.

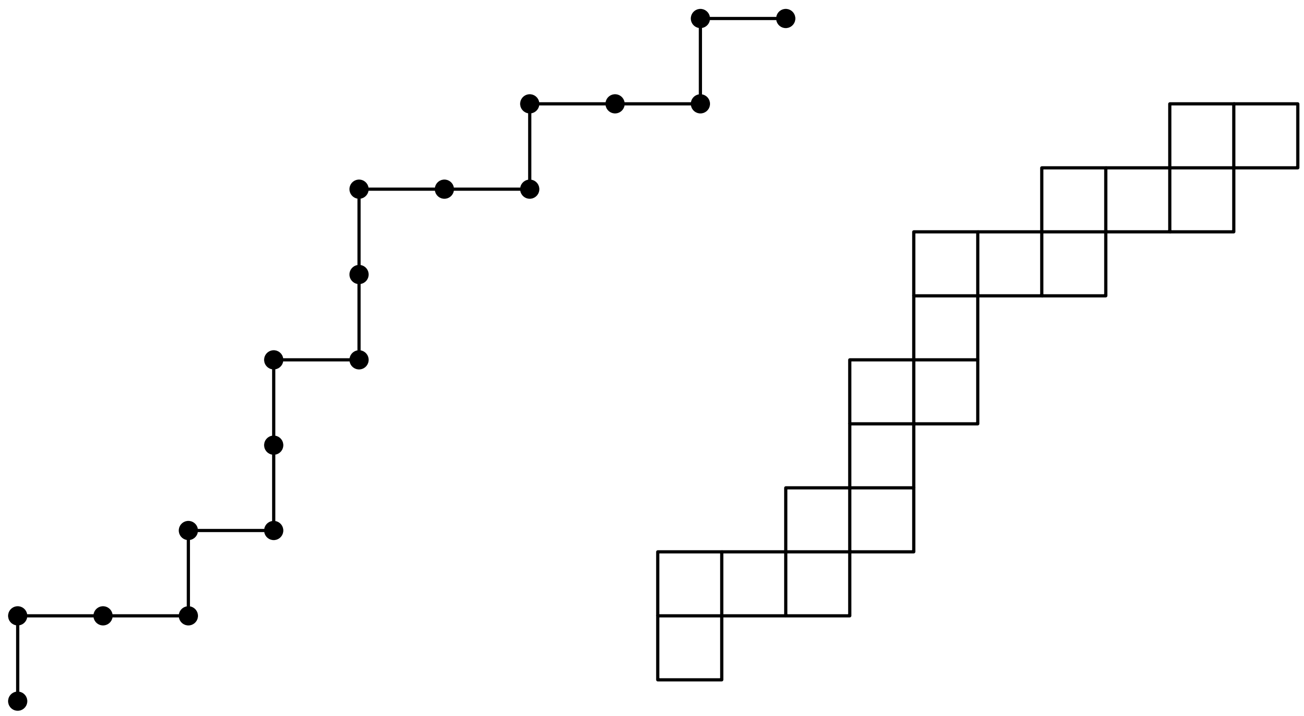
On the other hand, computing LR coefficients and the Kostka numbers efficiently is difficult (NP–complete). An alternative is to explore the equality among these coefficients, which motivates the study of equality of skew Schur functions for that

$$s_{\lambda/\mu} = s_{\rho/\sigma} \text{ if and only if } c^{\lambda}_{\mu\nu} = c^{\rho}_{\sigma\nu} \text{ for all } \nu.$$
$$s_{\lambda/\mu} = s_{\rho/\sigma} \text{ if and only if } K^{\nu}_{\lambda/\mu} = K^{\nu}_{\rho/\sigma} \text{ for all } \nu.$$

Definition. Two skew diagrams $\mathcal{D}$ and $\mathcal{E}$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if their corresponding skew Schur functions are equal, i.e., $s_{\mathcal{D}} = s_{\mathcal{E}}$.

Previous results

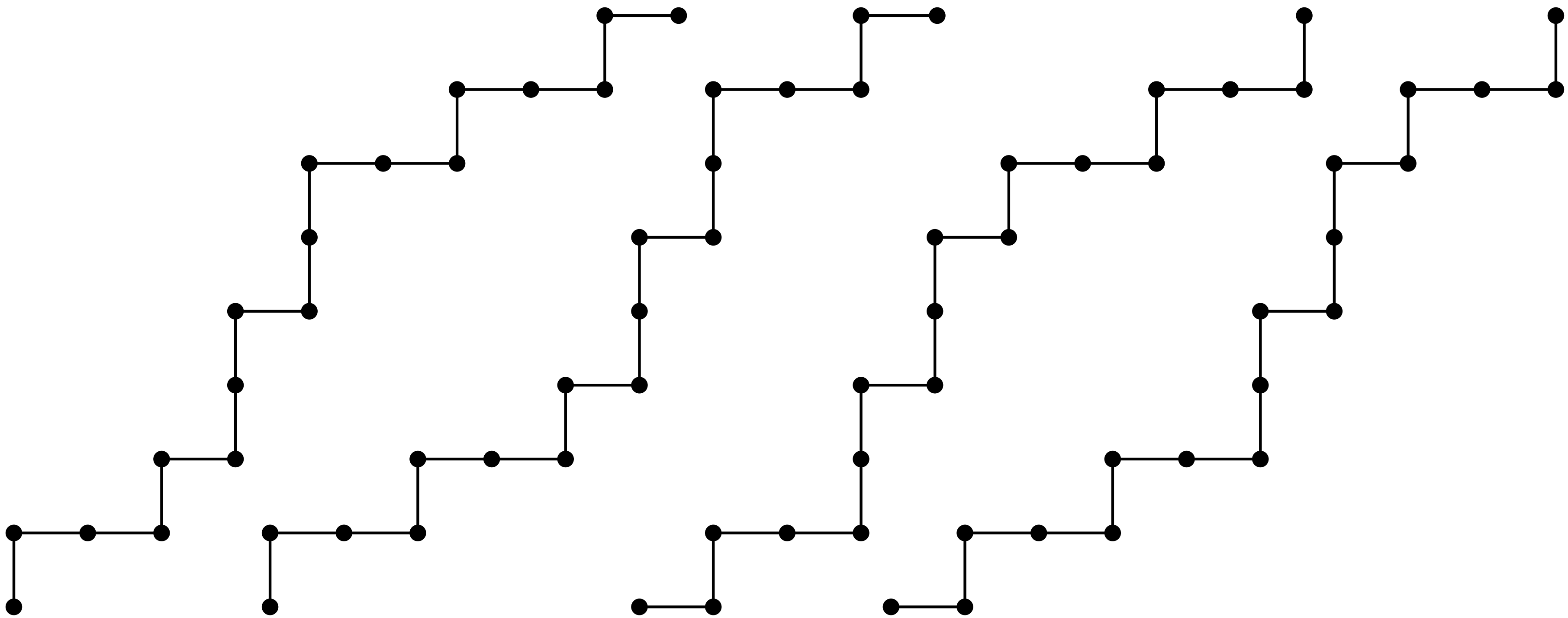
The study of equivalent skew diagrams was initiated by Billera, Thomas and van Willigenburg (2006) in which equivalence classes of ribbons are completely determined.



We use • to represent a box and two black dots are connected by an edge if the corresponding boxes are next to each other.

Previous results

Simply put, two ribbons are equivalent if and only if both ribbons admit parallel factorizations that differ by reversing some of the factors.



Each equivalence class contains 2^{κ} elements, where κ counts the occurrences of non-symmetric factors in the irreducible factorization.

Previous results

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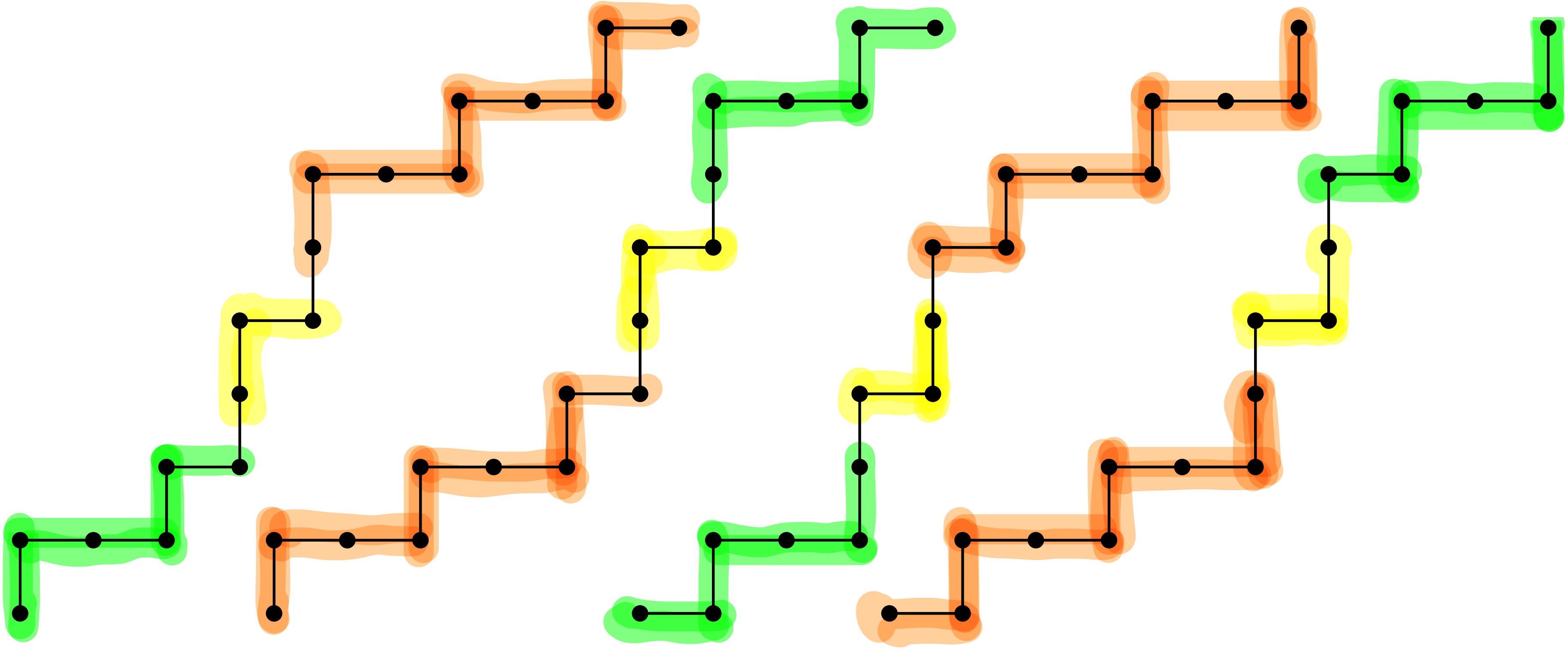
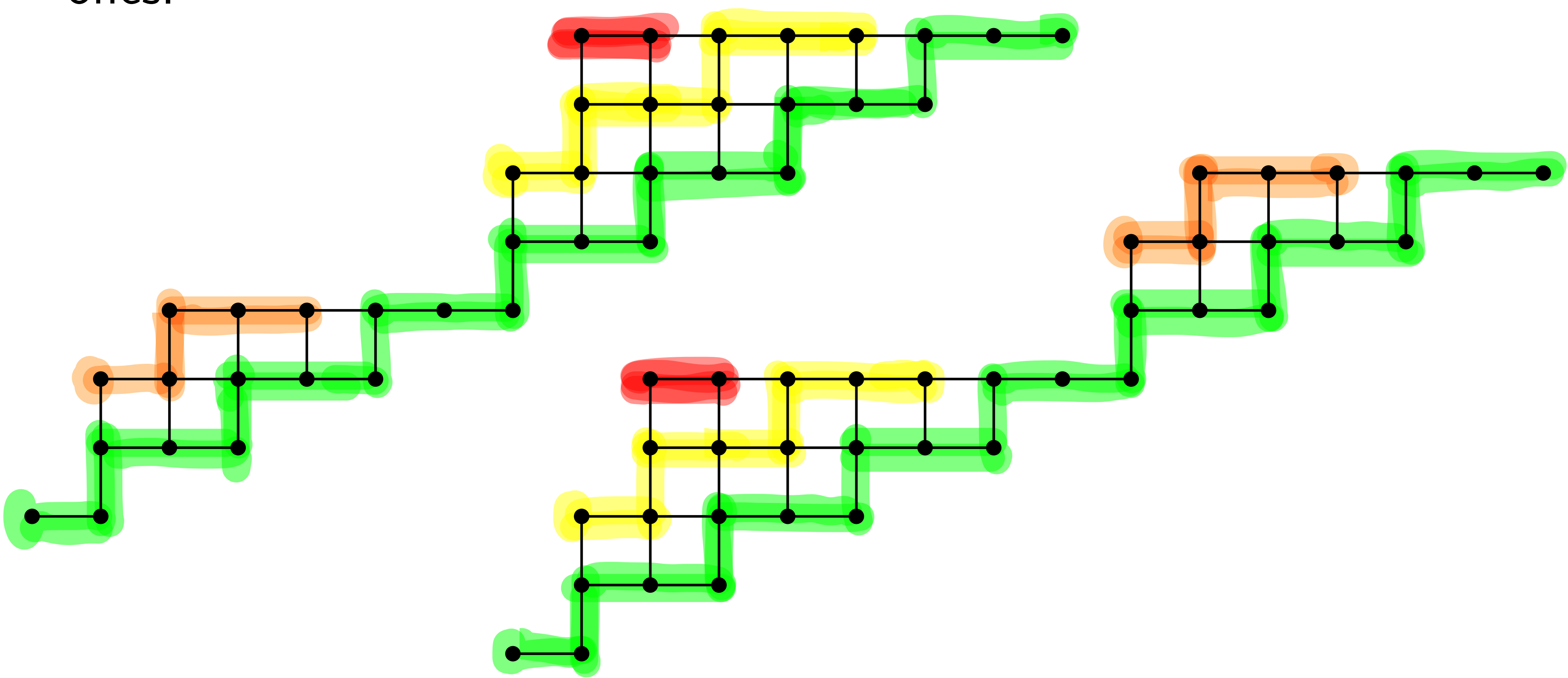


Each equivalence class contains 2^{κ} elements, where κ counts the occurrences of non-symmetric factors in the irreducible factorization.

Previous results

Subsequently, Reiner, Shaw and van Willigenburg (2007) introduced an operation between a ribbon and a skew diagram in order to construct larger equivalent non-ribbon skew diagrams from smaller ones.



Previous results

This operation was further extended to build up any skew diagram by McNamara and van Willigenburg (2009). They also established sufficient conditions for two skew diagrams to be equivalent.

Conjecture (McNamara and van Willigenburg 2009). Two skew diagrams $\mathcal{D}$ and $\mathcal{E}$ satisfy $\mathcal{D} \sim \mathcal{E}$ if and only if, for some r,

$$\mathcal{D} = (\cdots((\mathcal{E}_1 \circ_{W_2} \mathcal{E}_2) \circ_{W_3} \mathcal{E}_3) \cdots) \circ_{W_r} \mathcal{E}_r,$$
$$\mathcal{E} = (\cdots((\mathcal{E}'_1 \circ_{W'_2} \mathcal{E}'_2) \circ_{W'_3} \mathcal{E}'_3) \cdots) \circ_{W'_r} \mathcal{E}'_r,$$

- $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r$ are skew diagrams;
- for $2 \leq i \leq r$, $\mathcal{E}_i = W_i O_i W_i$ satisfies Hypotheses I–IV;
- $\mathcal{E}'_1 = \mathcal{E}_1$ or $\mathcal{E}'_1 = \mathcal{E}_1^*$;
- for $2 \leq i \leq r$, $(\mathcal{E}'_i, W'_i) = (\mathcal{E}_i, W_i)$ or $(\mathcal{E}'_i, W'_i) = (\mathcal{E}_i^*, W_i^*)$.

The equivalence class of $\mathcal{D}$ contains 2^{κ} elements, where κ counts factors $\mathcal{E}_i$ in any irreducible factorization of $\mathcal{D}$ such that $\mathcal{E}_i \neq \mathcal{E}_i^*$.

5. Equality on thickened ribbon Schur functions

Equality on thickened ribbon Schur functions

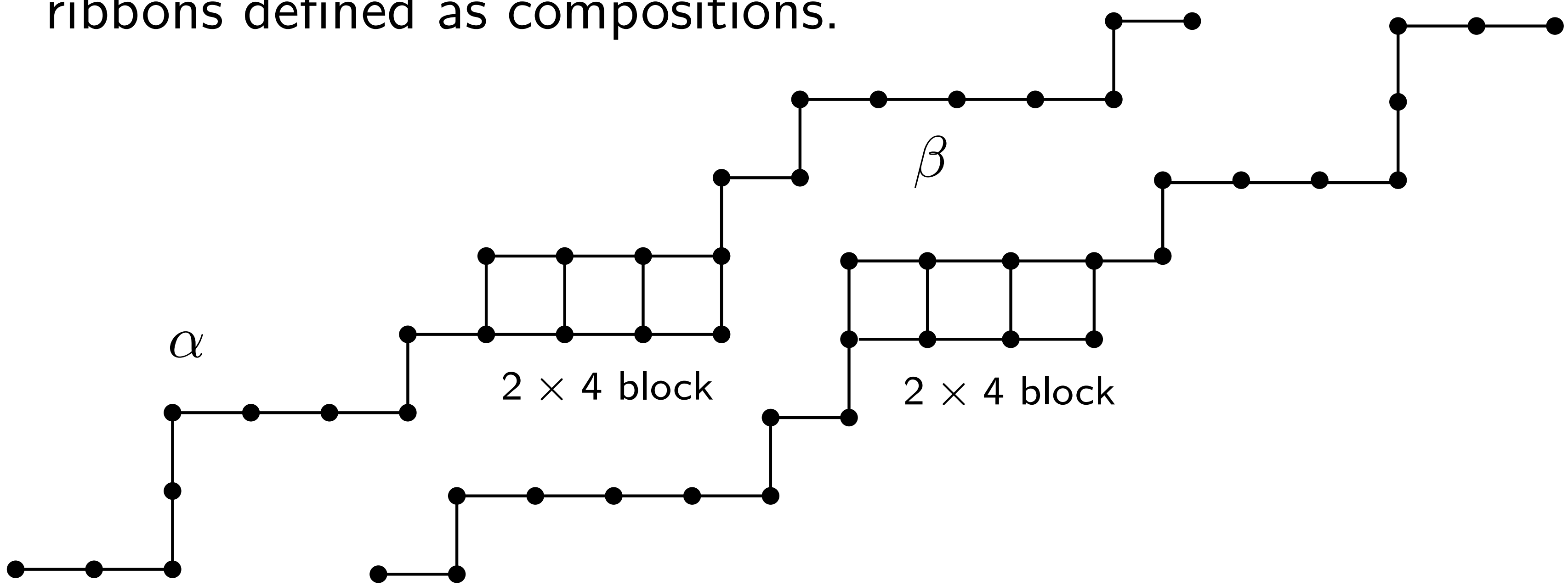
Theorem (J. and Li 2024). Two thickened ribbons $\mathcal{D}$ and $\mathcal{E}$ with exactly one $2 \times m$ or $m \times 2$ block for a fixed $m \geq 2$ are equivalent if and only if $\mathcal{D} = \mathcal{E}_1 \circ_{\mathcal{W}} \mathcal{E}_2$ and $\mathcal{E} = \mathcal{E}_1' \circ_{\mathcal{W}} \mathcal{E}_2'$ such that

- if $\mathcal{D}$ admits a nontrivial factorization, then $\mathcal{D} = \mathcal{E}_1 \circ_{\mathcal{W}} \mathcal{E}_2$ is a nontrivial factorization satisfying that $\mathcal{E}_1, \mathcal{E}_2$ are ribbons and $\mathcal{W} = (m-1)$ or $\mathcal{W} = (1^{m-1})$; otherwise $\mathcal{D}$ has only trivial factorization and $(\mathcal{E}_1, \mathcal{W}, \mathcal{E}_2) = (1, \emptyset, \mathcal{D})$;
- for $i = 1, 2$, $\mathcal{E}_i' = \mathcal{E}_i$ or $\mathcal{E}_i' = \mathcal{E}_i^*$.

The equivalence class of $\mathcal{D}$ contains 2^{κ} elements, where κ counts the occurrences of $\mathcal{E}_i \neq \mathcal{E}_i^*$ for $i = 1, 2$, independent of the choice of nontrivial factorizations of $\mathcal{D}$.

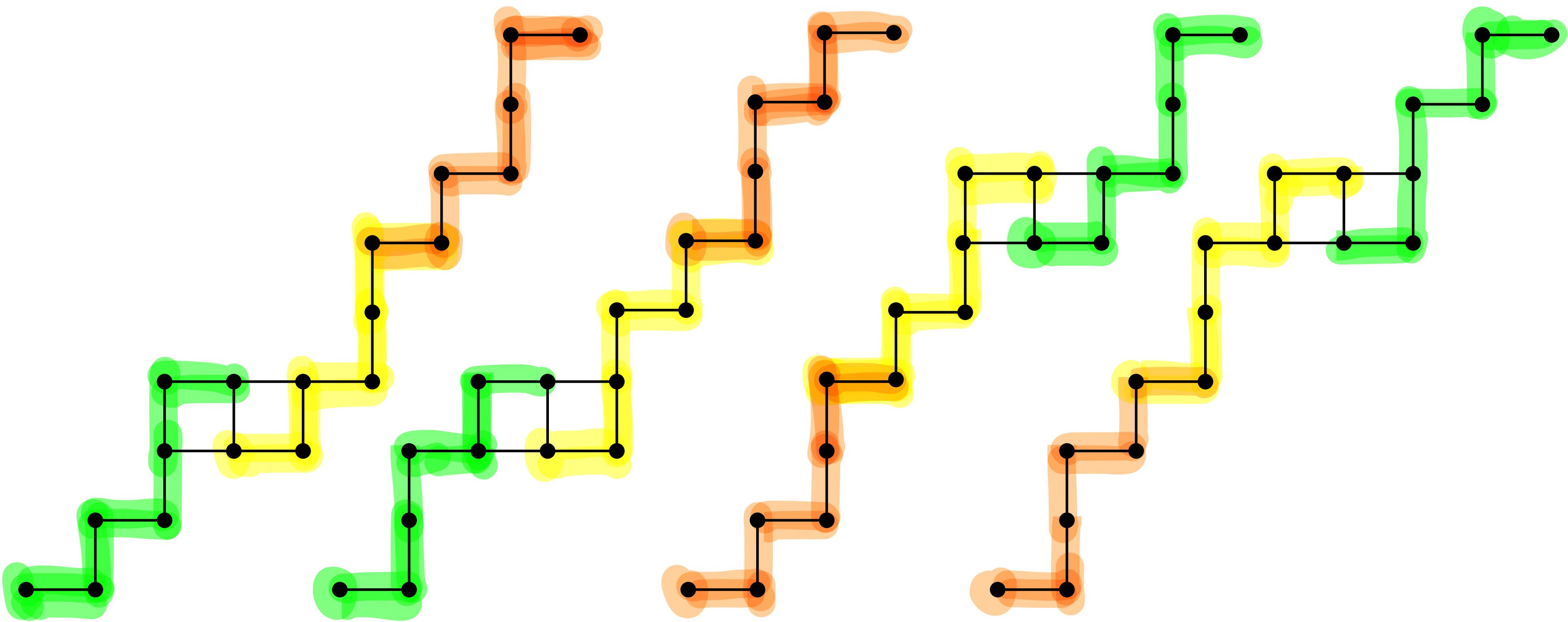
Equality on thickened ribbon Schur functions

Any thickened ribbon $\mathcal{D}$ with exactly one $2 \times m$ block is obtained by piling two ribbons, denoted by $\mathcal{D} = \alpha \boxdot \beta$ where α, β are ribbons defined as compositions.



For any $\mathcal{D}$, let $\mathcal{D}^*$ be the antipodal rotation of $\mathcal{D}$. When $|\alpha| = |\beta|$, our theorem shows that the equivalent class of $\mathcal{D}$ equals $\{\mathcal{D}, \mathcal{D}^*\}$.

Equality on thickened ribbon Schur functions



When $|\alpha| \neq |\beta|$, our theorem proves that the equivalent class of $\mathcal{D}$ equals $\{\mathcal{D}, \mathcal{D}^*\}$ if $\mathcal{D}$ is not periodic; equals $\{\mathcal{D}, \mathcal{D}^*, \mathcal{D}', (\mathcal{D}')^*\}$ otherwise, where $\mathcal{D} = \mathcal{S} \circ_{m-1} \mathcal{T}$ and $\mathcal{D}' = \mathcal{S} \circ_{m-1} \mathcal{T}^*$.

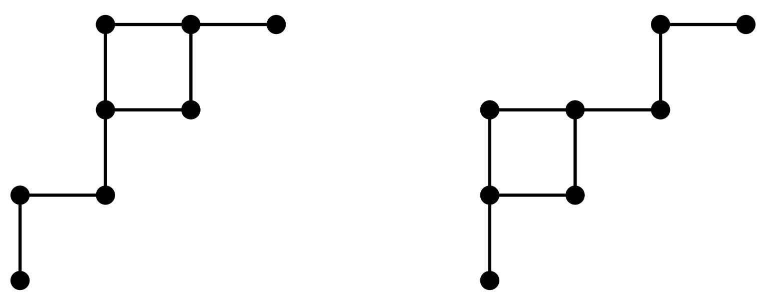
From ribbons to thickened ribbons

One major difficulty we have to overcome is the cancellations induced by the single $2 \times m$ block in the Jacobi-Trudi formula of skew Schur functions.

For a thickened ribbon $\mathcal{D}$ with one $2 \times m$ block of boxes,

$$s_{\mathcal{D}} = \det(h_{\lambda_i - \mu_j - i + j})_{i,j=1}^{n} = A - B$$

where the second term B is not zero. If $\mathcal{D}$ is a ribbon, then the second term B equals zero. The essential reason for this is that the ribbon Schur functions form a basis of the space of homogeneous symmetric function, whereas other skew shapes do not.



For example, given two thickened ribbons $\mathcal{D} = 122\boxdot 3$ and $\mathcal{E} = 12\boxdot 32$, we have $s_{\mathcal{D}} = s_{\mathcal{E}}$, but they have different A and B.

From ribbons to thickened ribbons

Essentially we get around this by classifying positive integers into equivalent classes and discuss in detail when cancellations exist.

Some questions for the future:

- Can we extend this approach to treat any fixed number of $2 \times m$ or $m \times 2$ blocks of boxes? Such thickened ribbons are also called double strips/ribbons in the book "Symmetric functions and Hall polynomials" by I.G. Macdonald.
- Can we make a detour (avoid any cancellation) to characterize equivalent connected skew diagrams? Both the conjecture and the theorem on ribbons state parallel factorizations, in addition to the four hypotheses.
- Study other equivalent symmetric functions such as skew Schur Q–functions, Lascoux–Leclerc–Thibon (LLT) polynomials, etc.

Thank you!

https://arxiv.org/pdf/2403.01843.pdf